Counting

Its Principles and Techniques (6)

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14. Pascal's Triangle

In Section 12 of [6], we established the Binomial Theorem (BT) which states that for all nonnegative integers n,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Let us display the binomial coefficients $\binom{n}{r}$ row by row following the increasing values of n as shown in Figure 14.1.

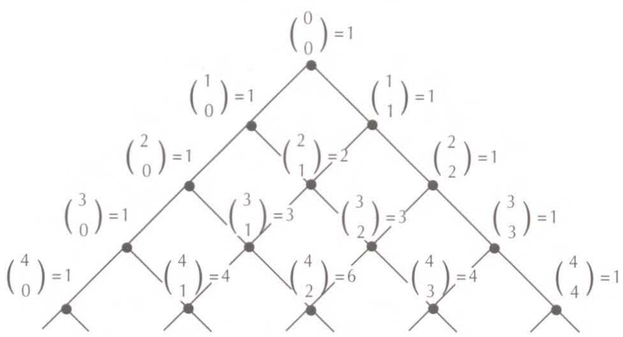

Figure 14.1

We observe that in Figure 14.1,

1. the binomial coefficient at a lattice point counts the number of shortest routes from the top lattice point (representing $\binom{0}{0}$) to the lattice point concerned. For example, there are $\binom{4}{2}$(=6) shortest routes from the lattice point representing $\binom{0}{0}$ to the lattice point $\binom{4}{2}$ (see Example 6.1 also);

2. the number pattern is symmetric with respect to the vertical line through the top lattice point, and this observation corresponds to the identity $\binom{n}{r} = \binom{n}{n-r}$;

3. Any binomial coefficient represented by an interior lattice point is equal to the sum of the two binomial coefficients represented by the lattice points on its 'shoulders' (see Figure 14.2). This observation corresponds to the identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$;

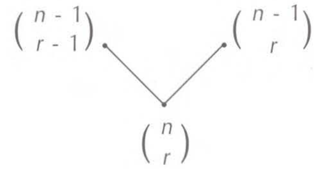

Figure 14.2

4. the sum of the binomial coefficients in the nth row is equal to 2^n and this fact corresponds to the identity

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n.$$

The number pattern of Figure 14.1 was known to Omar Khayyam and Jia Xian 贾宪 around 1100 A.D, and it was found in the book written by the Chinese mathematician Yang Hui 杨辉 in 1261 in which Yang Hui called it Jia Xian triangle. The number pattern in the form of Figure 14.3 was found in another book written by the Chinese mathematician Zhu Shijie 朱世杰 in 1303.

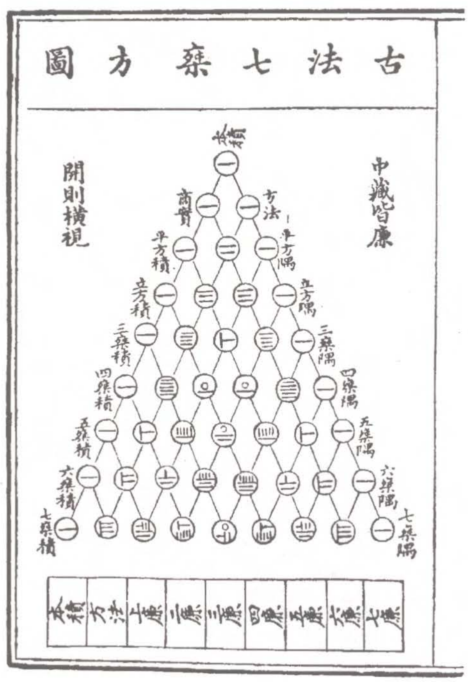

Figure 14.3

However, the number pattern of Figure 14.1 is generally called *Pascal's triangle* in memory of the great French mathematician Blaise Pascal (1623-1662) who also applied the 'triangle' to the study of *probability*, a subject dealing with 'chances'. For the history of this number pattern, readers are referred to the book [1].

Blaise Pascal

Figure 14.4

15. An Identity

Look at Pascal's triangle of Figure 15.1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 15.1

What is the sum of the six binomial coefficients enclosed in the rectangle? The answer is '56'. Note that this answer appears as another binomial coefficient located at the right side of '21' next row. Is this situation just a coincidence? Let us take a closer look.

Observe that

$$
\begin{aligned}
&1 + 3 + 6 + 10 + 15 + 21 \\
&= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} \\
&= \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} \quad \left(\text{as } \binom{2}{2} = \binom{3}{3}\right) \\
&= \binom{4}{3} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} \\
&= \binom{5}{3} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2} \\
&= \binom{6}{3} + \binom{6}{2} + \binom{7}{2} \\
&= \binom{7}{3} + \binom{7}{2} \\
&= \binom{8}{3} (=56),
\end{aligned}
$$

by applying the identity $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$.

The above result is really a special case of a general situation. As a matter of fact, the above argument could also be used to establish the following general result:

For any nonnegative integers n and k with $n \geq k$,

$$\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1} \qquad \text{(B4)}$$

(see Figure 15.2).

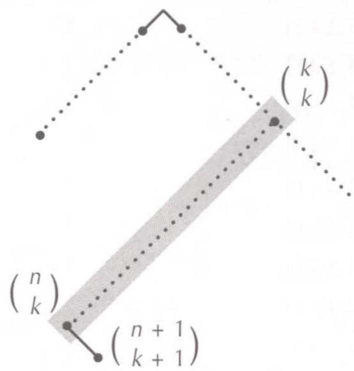

Figure 15.2

By the symmetry of Pascal's triangle, one obtains the following accompanied identity of (B4) (see also Figure 15.3):

$$\binom{k}{0} + \binom{k+1}{1} + \dots + \binom{k+m}{m} = \binom{k+m+1}{m}. \qquad \text{(B5)}$$

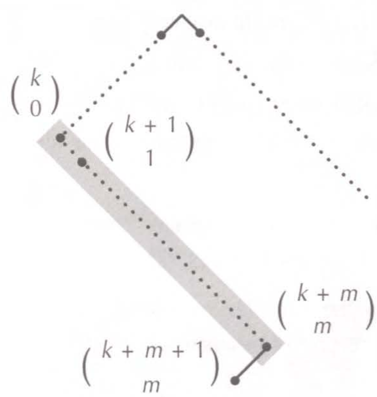

Figure 15.3

16. An IMO Problem

In this section, we show an application of identity (B4) in the solution of the following problem, which appeared in International Mathematical Olympiad 1981.

Example 16.1 Let $1 \leq r \leq n$ and consider all r-element subsets of the set $\{1, 2, \dots, n\}$. Each of these subsets has a smallest member. Let $F(n,r)$ denote the arithmetic mean of these smallest numbers. Prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

As an illustration of this problem, consider the case when $n = 6$ and $r = 4$. There are $\binom{6}{4}$(=15) 4-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$. They and their smallest members are listed in Table 16.1. By definition of $F(6,4)$, we have

$$F(6, 4) = (10.1 + 4.2 + 1.3) \div 15 = \frac{7}{5},$$

and this is equal to $\frac{n+1}{r+1}$ when $n = 6$ and $r = 4$.

4-element subsets of {1,2,...,6}	Smallest members
{1,2,3,4}	1
{1,2,3,5}	1
{1,2,3,6}	1
{1,2,4,5}	1
{1,2,4,6}	1
{1,2,5,6}	1
{1,3,4,5}	1
{1,3,4,6}	1
{1,3,5,6}	1
{1,4,5,6}	1
{2,3,4,5}	2
{2,3,4,6}	2
{2,3,5,6}	2
{2,4,5,6}	2
{3,4,5,6}	3

Table 16.1

Write $N_n = \{1, 2, ..., n\}$. To evaluate $F(n,r)$, it is clear that we need to find out first

1. which numbers in N_n could be the smallest member of an r-element subset of N_n (in the above example, these are 1, 2, 3 but not 4, 5, 6), and
2. how many times such a smallest member occurs (in the above example, '1' occurs 10 times, '2' 4 times and '3' once);

and then sum these smallest members up, and finally divide the sum by $\binom{n}{r}$, the number of r-element subsets of N_n, to obtain the 'average'.

The last r elements (according to the magnitude) of the set N_n are:

$$\underbrace{n - r + 1,\ n - r + 2,\ ...,\ n - r + r (= n)}_{r}$$

It follows that '$n - r + 1$' is the *largest* possible number to be the smallest member of an r-element subset of N_n. Hence 1, 2, 3, ..., $n - r + 1$ are all possible numbers which could be 'smallest members'.

Let $k \in \{1, 2, 3, ..., n - r + 1\}$. Our next task is to find out how many times 'k' occurs as the 'smallest member'. To form an r-element subset of N_n containing 'k' as the smallest member we simply form a $(r - 1)$–element subset from the $(n - k)$–element set $\{k + 1, k + 2, ..., n\}$ (and then add 'k' to it). The number of $(r - 1)$–element subsets of $\{k + 1, k + 2, ..., n\}$ is given by $\binom{n-k}{r-1}$. Thus, 'k' occurs $\binom{n-k}{r-1}$ times as the 'smallest member'. Let Σ denote the sum of all these 'smallest members'. Then, as $k = 1, 2, ..., n - r + 1$, we have

$$\Sigma = 1\binom{n-1}{r-1} + 2\binom{n-2}{r-1} + 3\binom{n-3}{r-1} + ... + (n-r+1)\binom{n-(n-r+1)}{r-1}$$

$$= (n-r+1)\binom{r-1}{r-1} + ... + 3\binom{n-3}{r-1} + 2\binom{n-2}{r-1} + 1\binom{n-1}{r-1}$$

$$\left.\begin{aligned} = &\binom{r-1}{r-1} + ... + \binom{n-3}{r-1} + \binom{n-2}{r-1} + \binom{n-1}{r-1} \\ + &\binom{r-1}{r-1} + ... + \binom{n-3}{r-1} + \binom{n-2}{r-1} \\ + &\binom{r-1}{r-1} + ... + \binom{n-3}{r-1} \\ &\vdots \\ + &\binom{r-1}{r-1} \end{aligned}\right\} n - r + 1 \text{ rows of summands}$$

Now by applying (B4) to each summand above except the last one and noting that $\binom{r-1}{r-1} = \binom{r}{r}$, Σ can be simplified as

$$\Sigma = \underbrace{\binom{n}{r} + \binom{n-1}{r} + ... + \binom{r}{r}}_{n-r+1}$$

By applying (B4) once again, we have

$$\Sigma = \binom{n+1}{r+1}.$$

Finally, by definition of $F(n, r)$, it follows that

$$F(n, r) = \Sigma \div \binom{n}{r} = \binom{n+1}{r+1} \div \binom{n}{r}$$

$$= \frac{(n+1)!}{(r+1)!(n-r)!} \cdot \frac{r!(n-r)!}{n!}$$

$$= \frac{n+1}{r+1}$$

as desired. M²

References

[1] A. W. F. Edwards, Pascal's Arithmetical Triangle, Oxford University Press, 1987.

[2] K. M. Koh and B. P. Tan, Counting–Its Principles and Techniques (1), Mathematical Medley Vol. 22 March (1995) 8 – 13.

[3] K. M. Koh and B. P. Tan, Counting–Its Principles and Techniques (2), Mathematical Medley Vol. 22 September (1995) 47 – 51.

[4] K. M. Koh and B. P. Tan, Counting–Its Principles and Techniques (3), Mathematical Medley Vol. 23 March (1996) 9 – 14.

[5] K. M. Koh and B. P. Tan, Counting–Its Principles and Techniques (4), Mathematical Medley Vol. 23 September (1996) 44 – 50

[6] K. M. Koh and B. P. Tan, Counting–Its Principles and Techniques (5), Mathematical Medley Vol. 24 March (1997) 8 – 12