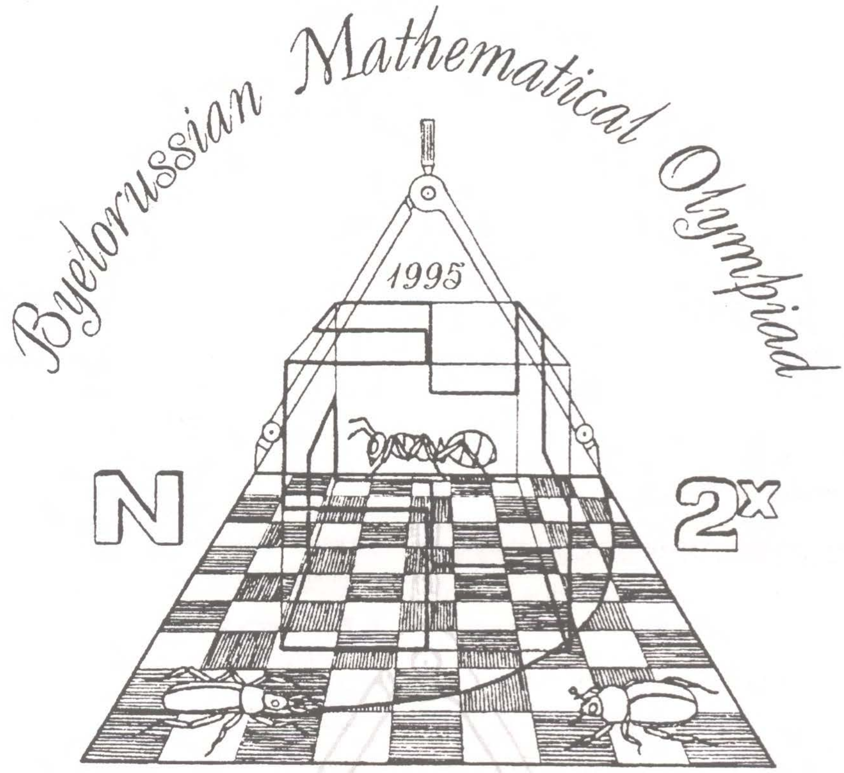
Competition Corner

by Tay Tiong Seng

XLV



All communications concerning this column should be sent to *Dr. Tay Tiong Seng, Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260.*

In this issue we publish the problems of the XLV Byelorussian Mathematical Olympiads held during 1994/95. The structure of the Byelorussian Olympiads is as follows. There are four rounds: The first is the school round in October followed by the district round in November. The third is the regional round and Minsk Mathematical Olympiad in January. The fourth, which is the final round, is held in March-April. The 3rd and 4th rounds are over two days with usually 4 problems with a time limit of 4 hours on each day. The participants are divided into four categories in accordance with their level in schools: Category A–11 form (16-17 years); Category B–10 form (15-16 years); Category C–9 form (14-15 years); Category D–8 form (13-14 years). Byelorussia first participated in the IMO in 1992 as an observer after the break up of the former Soviet Union. In the 38th IMO, its contestants won 2 silver and 3 bronze medals. Finally we discuss solutions sent in by readers to problems of the 13th Iranian Mathematical Olympiad featured in the last issue. Readers are urged to send in their solutions to the problems featured here as well as the problems of the 38th IMO (see p. 75). All contributors with correct solutions will be acknowledged.

XLV Byelorussian Mathematical Olympiads—Final Round

Category D

1. Mark six points in a plane so that no three of them are collinear and any three of them are the vertices of an isosceles triangle.

2. Find all positive integers n such that both n and $n + 100$ have an odd number of divisors.

3. Some students of a group were friends of some others. One day all students of the group took part in a picnic. During the picnic some friends had a quarrel with each other, but some other students that had not been friends before became friends. After the picnic the number of friends of each student was changed by one. Prove that the number of students in the group is even.

4. Given a triangle ABC, let K be the midpoint of side AB and L be a point on AC such that $AL = LC + CB$. Prove that $\angle KLB = 90°$ if and only if $AC = 3CB$.

5. Two circles touch at a point M and lie inside a rectangle $ABCD$. It is known that one of them touches the sides AB and AD, and the other touches the sides AD, BC and CD. The second circle has radius four times as long as the radius of the first one. Find the ratio in which the common tangent of the circles that passes through M divides the sides AB and AD.

6. Let p and q be distinct positive integers. Prove that at least one of the equations

$$x^2 + px + q = 0 \text{ or } x^2 + qx + p = 0$$

has a real root.

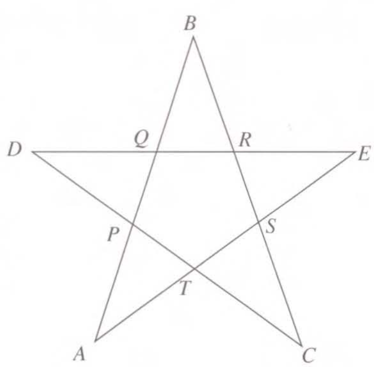
7. The expression 1*2*3*4*5*6*7*8*9 is written on a blackboard. Two boys, Bill and Peter, play the following game. They replace a "*" by a "+" (addition) or by a "x" (multiplication). They make their moves in turn, one of them may use only the sign "+", while the other only the sign "x". At the beginning, Bill selects the sign he will use, and then defines who will make the first move. Bill tries to attain the result to be an even number, while Peter wants the result to be an odd number. Prove that Peter can always win.

8. Five numbers 1, 2, 3, 4, 5 are written on a blackboard. At each step one may erase any two numbers a and b and write the numbers $a + b$ and $a \cdot b$. Can the numbers 21, 27, 64, 180, 540 appear on the blackboard after a finite number of such steps?

Category C

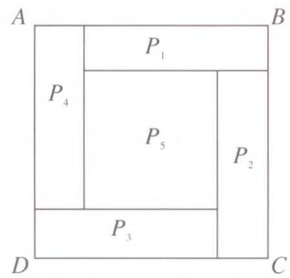
1. Six distinct numbers $n_1, n_2, n_3, n_4, n_5, n_6$ are given. For each two of these numbers, Bill calculates their sum. What is the largest possible number of distinct primes among the sums obtained by Bill?

2. The figure shows a closed polygon $ABCDEA$, and the following equalities hold: $AP = QB$, $BR = SC$, $CT = PD$, $DQ = RE$, where P, Q, R, S, T are the self-crossing points. Prove that $ES = TA$.



3. On an island, each islander either always tells the truth or always lies. A new governor wants to determine who is a liar on the island and who is a truth-teller. For this purpose, every day he gathers a group of the islanders and asks each of them how many liars there are in the group. He writes down the names of the islanders and their answers in a notebook for future analysis. What is the minimum number of days the governor would need to implement his plan if he knows that not all the islanders are liars? (It is assumed that each islander knows who is who on the island.)

4. A rectangle $ABCD$ is partitioned into five rectangles P_1, P_2, P_3, P_4, P_5 as shown in the figure. It is known that P_5 is a square, and the areas of P_1, P_2, P_3 and P_4 are equal. Prove that $ABCD$ is a square.



5. Let AB be the diameter of a semicircle. A point M is marked on the semicircle, and a point K is marked on AB. A circle with centre P passes through A, M, K and a circle with centre Q passes through M, K, B. Prove that M, K, P and Q lie on the same circle.

6. Three parabolas are defined by the functions $y = f(x)$, $y = g(x)$ and $y = h(x)$. The branches of all parabolas are directed upwards. Let a, b, c be the x-coordinates of the vertices of the three parabolas, respectively. It is known that $f(b) < f(c)$ and $g(c) < g(a)$. Prove that $h(b) < h(a)$.

7. Two towns A and B are connected by a straight road. The cyclists start from A to B one after another at 8:00am, with equal time intervals, and move with equal and constant speeds. The motor-cyclists start from B to A one after another at 8:00am, with equal time intervals, and move with equal and constant speeds. The last cyclist reaches B at 4:00pm, and the last motor-cyclist reaches A at 4:00pm. Let M denote the middle point of AB; let X and Y be the numbers of meetings of the cyclists and the motor cyclists since 8:00am till 12:00 noon, and from A to M, respectively. Compare X and Y. (It is supposed that there is no meeting at 12:00 sharp and no meetings at M.)

8. Sixty five beetles are placed at some 65 cells of a 9 x 9 square board. At each move, every beetle creeps to an adjacent cell. No beetle makes two horizontal moves or two vertical moves in succession. Prove that after some moves at least two beetles are at the same cell.

Category B

1. A point B is marked inside a regular hexagon $A_1A_2A_3A_4A_5A_6$ so that $\angle A_2A_1B = \angle A_4A_3B$. Find $\angle A_1A_2B$.

2. Find the product of three distinct real numbers a, b, c if they satisfy the system of equations

$$a^3 = 3b^2 + 3c^2 - 25,$$
$$b^3 = 3c^2 + 3a^2 - 25,$$
$$c^3 = 3a^2 + 3b^2 - 25,$$

3. "Words" are formed with the letters A and B. Using the words x_1, x_2, ..., x_n we can form a new word if we write these words consecutively one next to another: $x_1x_2...x_n$. A word is called a palindrome, if it is not changed after rewriting its letters in the reverse order. Prove that any word with 1995 letters A and B can be formed with less than 800 palindromes.

4. Find all functions f, $f: \boldsymbol{R} \to \boldsymbol{R}$, satisfying the equality

$$f(f(x - y)) = f(x) - f(y) + f(x)f(y) - xy$$

for all x and y.

5. Let AK, BL and CM be the altitudes of an acute angled triangle ABC. Prove that if $9\overrightarrow{AK} + 4\overrightarrow{BL} + 7\overrightarrow{CM} = 0$, the zero vector, then there is an angle in ABC that is equal to 60°

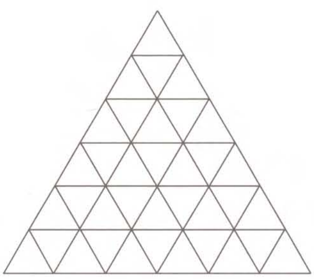
6. Given three real numbers such that the sum of any two of them is not equal to 1, prove that there are two numbers x and y such that $xy/(x + y - 1)$ does not belong to the interval (0,1).

7. Let $\boldsymbol{Q}^*$ denote the set of rational numbers, each greater than 1.

(a) Is it possible to partition $\boldsymbol{Q}^*$ into two disjoint sets A and B so that the sum of any two numbers from A belongs to A and the sum of any two numbers from B belongs to B?

(b) Is it possible to partition $\boldsymbol{Q}^*$ into two disjoint sets A and B so that the product of any two numbers from A belongs to A and the product of any two numbers from B belongs to B?

8(a). Each side of an equilateral triangle is divided into 6 equal parts; the points of this partition are connected by lines parallel to the sides of triangle. Each vertex of the obtained triangular grid is occupied by exactly one beetle. All beetles begin crawling along the links of the grid simultaneously with the same speed. The beetles creep according to the following rule: when a beetle reaches a vertex of the grid it must turn (to the right or to the left) by 60° or 120°. (The beetles do not turn back at any point.) Prove that at some moment two beetles meet at a vertex of the grid.



(b). Would the statement remain true if each side of the triangle is divided into 5 equal parts?

Category A

1. There are 20 rooms in a hotel on a sea beach. The building of the hotel has only one storey and all rooms are arranged along one side of the common corridor. The rooms are numbered by the integers from 1 to 20 consecutively. A visitor may rent either one room for two days or two neighbouring rooms for one day. The cost of a room is \$1 per day. The sea-bathing season lasts 100 days. It is known that room No. 1 was rented at the first day and room No. 20 was not rented at the last day of the season. Prove that owners of the hotel receive at most \$1996 during the season.

2(a). After a lesson in mathematics, the Ox-axis and the graph of the function $y = 2^x$ were left on a blackboard but the Oy-axis and the scale were erased. Give a Euclidean construction (using a straight edge and compasses only) for the Oy-axis and the unit of the scale.

(b). Give an Euclidean construction of both axes and the unit of the scale if both axes and the scale were erased but the graph of $y = 2^x$ and a straight line parallel to the Ox-axis were left on the blackboard.

3. Find all functions f, $f: \boldsymbol{R} \to \boldsymbol{R}$, satisfying the equality

$$f(f(x + y)) = f(x + y) + f(x)f(y) - xy$$

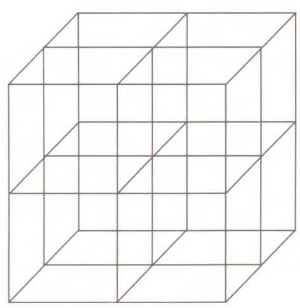
for all real x and y.

4. Given a triangle ABC with $\angle ABC = 3\angle CAB$, let M and N be chosen on side CA so that $\angle CBM = \angle MBN = \angle NBA$. Suppose that X is an arbitrary point on BC. If L is the intersection point of AX and BN, and K is the intersection point of NX and BM, prove that the lines KL and AC are parallel.

5. The centre O_1 of circle S_1 lies on a circle S_2 with the centre O_2. The radius of S_2 is greater than that of S_1. Let A be the intersection point of S_1 and O_1O_2. Consider a circle S centred at an arbitrary point X on S_2 and passing through A; let Y be the intersection point of S and S_2 (different from A). Prove that all lines XY are concurrent, as X runs along S_2.

6. Given real numbers a and b, such that the cubic polynomial $x^3 + \sqrt{3}(a - 1)x^2 - 6ax + b$ has three real roots, prove that $|b| \leq |a + 1|^3$.

7. The lattice frame construction of a 2 x 2 x 2 cube is formed with 54 metal shafts of length 1 (points of shafts' connection are called junctions). An ant starts from some junction A and creeps along the shafts in accordance with the following rule: when the ant reaches the next junction it turns to the perpendicular shaft. At some moment the ant reaches the initial junction A; there is no junction (except for A) where the ant has been twice. What is the maximum length of the ant's path?



8. Is it possible to partition the set of all rational numbers into two disjoint subsets A and B so that

(a) the sum of any two numbers from A, as well the sum of any two numbers from B, belongs to A?

(b) the sum of any two distinct numbers from A, as well as the sum of any two distinct numbers from B, belongs to A?

Solutions of the 13th Iranian National Mathematics Olympiad

1. Show that for every $n > 3$ there exist two sets of integers $A = \{x_1, x_2, \ldots, x_n\}$, $B = \{y_1, y_2, \ldots, y_n\}$ such that:

   (i) A and B are disjoint.

   (ii) $x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n$.

   (iii) $x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2$.

Solved by Thevendran Senkodan, Raffles Junior College who noted that the result is also true for n = 3, and Jeffrey Pang Chin How, Hwa Chong Junior College. The solution presented here follows basically that of Jeffrey.

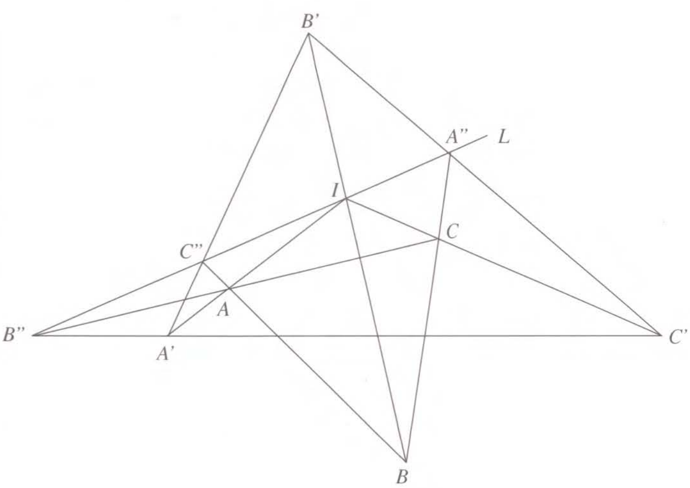
First we note that $A = \{x_1, x_2, \ldots x_n\}$ and $B = \{y_1, y_2, \ldots y_n\}$ are a solution if and only if

$$\sum_{i=1}^{n}(x_i - y_i) = 0 \text{ and } \sum_{i=1}^{n}(x_i^2 - y_i^2) = \sum_{i=1}^{n}(x_i - y_i)(x_i + y_i) = 0$$

Thus if $x_i + y_i = k$ for $i = 1, 2, \ldots, n$, then $A' = \{x_1 + d, x_2 + d, \ldots, x_n + d\}$ and $B' = \{y_1 + d, y_2 + d, \ldots, y_n + d\}$ are also a solution. If we now take $k = 0$ and $a_n = n(n+1)/2$, then $A_0 = \{1, 2, \ldots, n, -a_n\}$, $B_0 = \{-1, -2, \ldots, -n, a_n\}$ are a solution. Then $A_0' = \{1 + m, 2 + m, \ldots, n + m, -a_n + m\}$, $B_0' = \{-1 + m, -2 + m, \ldots, -n + m, a_n + m\}$ also form a solution and by taking $m > a_n$, we can have a solution where all the integers are positive even though this is not required in the problem.

2. ABC is an acute angled triangle and L is a line in its plane. The line L is reflected about each of the sides of ABC. The three resulting lines intersect pairwise to form a triangle $A'B'C'$. Prove that the incenter of $\Delta A'B'C'$ lies on the circumcircle of ΔABC.

Solved by Thevendran Senkodan, Raffles Junior College and Jeffrey Pang Chin How, Hwa Chong Junior College. The solution presented is due to Senkodan.



Let A', B', C' be labeled so that $A'B'$, $B'C'$ and $A'C'$ are, respectively, the reflections of L about AB, BC and AC. Since the reflection of L about AB yields $A'B'$, A is equidistant from L and $A'B'$. Similarly, A is also equidistant from L and $A'C'$. Thus AA' is the bisector of $\angle B'A'C'$. Similarly, BB' and CC' are the bisectors of $\angle A'B'C'$ and $\angle A'C'B'$. Thus the incenter I of $A'B'C'$ is the intersection point of AA', BB' and CC'. Now L intersects with at least two of the sides of ABC. Suppose L intersects BC and AC at A'' and B'', respectively. Triangles $C'B'A'$ and $C'A''B''$ have a common angle at C'. Thus

$$\angle C'A'B' + \angle C'B'A' = \angle C'A''B'' + \angle C'B''A''$$

Therefore

$$\begin{aligned}\angle ACB &= \angle CA''B'' + \angle CB''A'' \\ &= 1/2(\angle C'A''B'' + \angle C'B''A'') \\ &= 1/2(\angle C'A'B' + \angle C'B'A') = \angle AIB\end{aligned}$$

Hence I lies on the circumcircle of triangle ABC.

4. Let $S = \{2^m3^n | m, n \text{ are nonnegative integers}\}$. Prove that every natural number can be written as a sum of distinct elements of S such that none of them is a multiple of another.

Solved by Thevendran Senkodan, Raffles Junior College.

First note that $1 = 2^03^0$, $2 = 2^13^0$, $3 = 2^03^1$, $4 = 2^23^0$, $5 = 2^13^0 + 2^03^1$, can be expressed in the desired manner. The proof is by induction. Suppose it is now possible to express every positive integer up to k, $k \geq 5$, in the desired manner. Consider $k + 1$. Let $k + 1 = 2^a3^bp$, where a and b are nonnegative integers and p is coprime with 2 and 3. If either a or b is positive, then $p \leq k$, and the result follows by induction. Now suppose that $a = b = 0$ and $p = k + 1$. There is some positive integer m such that $3^{m+1} > p > 3^m$ (take for example the largest m for which 3^m is less than p). Let $q = p - 3^m$. Then $2\cdot3^m > q$. Since q is even and is less than $k + 1$, it can be expressed in the desired form:

$$q = s_1 + s_2 + \ldots + s_n$$

where s_n is even for each $i = 1, 2, \ldots, n$. Since $s_i < 2\cdot3^m$ for each $i = 1, 2, \ldots, n$,

$$p = s_1 + s_2 + \ldots + s_n + 3^m$$

is of the desired form.

5. Prove that for every positive integer n

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [3\sqrt{n+1}]$$

where $[x]$ denotes the smallest integer greater than or equal to x.

Solved by Thevendran Senkodan, Raffles Junior College and Fred P. F. Leung, National University of Singapore, who proved a stronger result

$$[\sqrt{9n+8}] = [\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [3\sqrt{n+1}]$$

for all positive integers n. The solution given below is a combination of the solutions by them.

Note that for all positive integers n,

$$\begin{aligned}&(\sqrt{n} + \sqrt{n+1} + \sqrt{n+2})^2 \\ &= 3n + 3 + 2(\sqrt{n(n+1)} + \sqrt{n(n+2)} + \sqrt{(n+1)(n+2)}) \\ &< 9n + 9\end{aligned}$$

where the last inequality follows from the AM-GM inequality. Thus

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] \leq [\sqrt{9n+9}].$$

Since

$$n(n+1) > n^2 + \frac{n}{2} + \frac{1}{16} = (n + \frac{1}{4})^2,$$

$$n(n+2) > n^2 + n + \frac{1}{4} = (n + \frac{1}{2})^2 \text{ and}$$

$$(n+1)(n+2) > (n+\tfrac{5}{4})^2$$

we have

$$2(\sqrt{n(n+1)} + \sqrt{n(n+2)} + \sqrt{(n+1)(n+2)}) > 6n+4$$

and whence

$$\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} > \sqrt{9n+7}.$$

If $9n+7$ is a square, then

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] \geq \sqrt{9n+7} + 1 = [\sqrt{9n+7}] + 1 = [\sqrt{9n+8}].$$

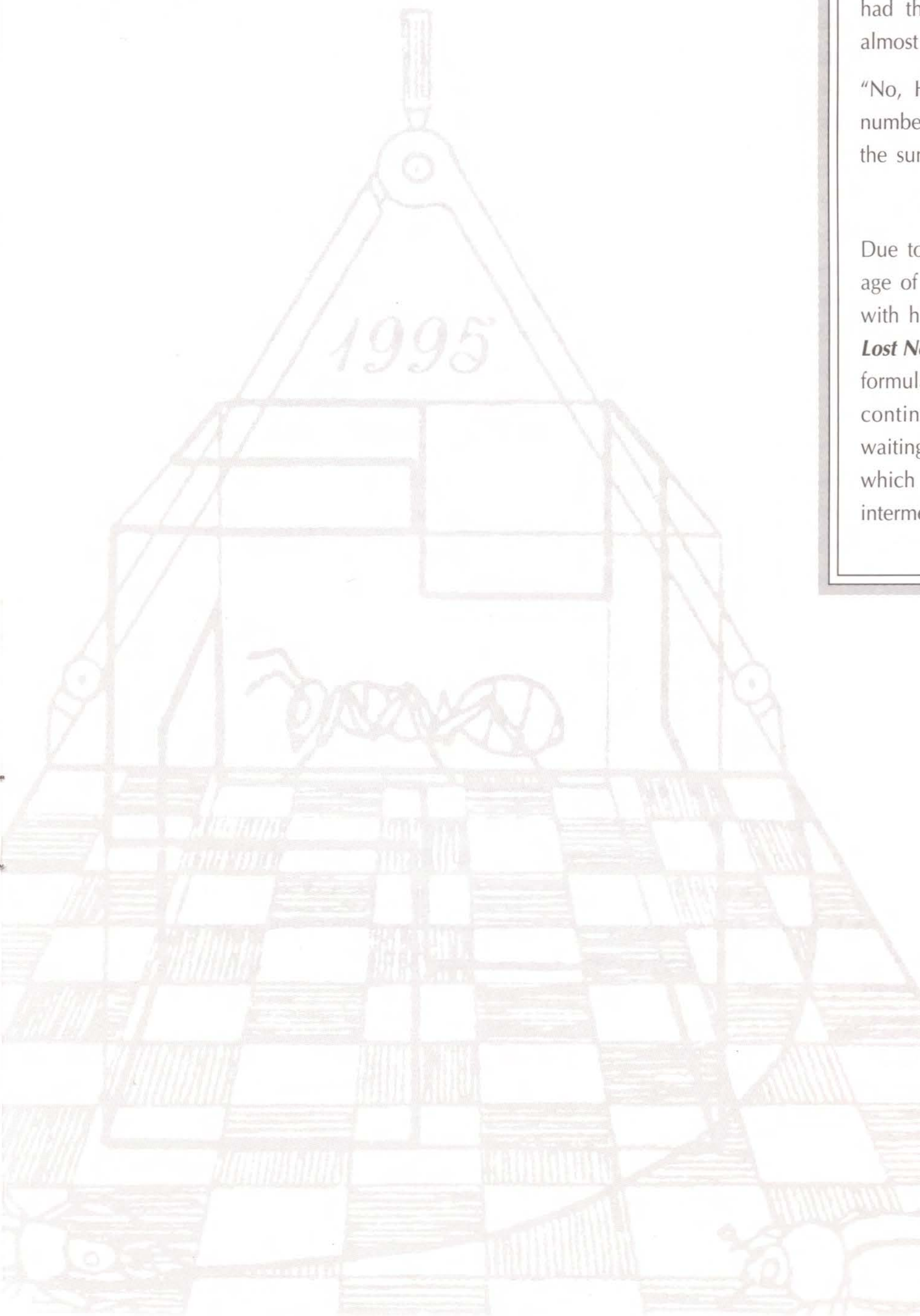
If $9n+7$ is not a square, then

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] \geq \sqrt{9n+7} = [\sqrt{9n+8}].$$

for if there is a positive integer k such that $\sqrt{9n+7} < k \leq \sqrt{9n+8}$, then $9n+8 = k^2$, and $k^2 \equiv 2 \pmod 3$ which is impossible. The fact that $[\sqrt{9n+8}] = [\sqrt{9n+9}]$ can be established similarly. Thus

$$[\sqrt{9n+8}] = [\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [3\sqrt{n+1}].$$

—Don't forget to send in your nice solutions and competition problems—



A Mathematical Mystic

contributed by Yan Kow Cheong

Recognising the self-educated Indian mathematician's genius, the great English mathematician **Godfrey H. Hardy** (1877-1947) invited **Srinivasa Ramanujan** (1887-1920) to Cambridge to pursue his love and thirst for mathematics. Once when visiting his *protégé* at the hospital, Hardy mentioned that the taxi which had taken him to the hospital had the dull number 1729. Ramanujan reacted almost immediately,

"No, Hardy! No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."

$$(9^3 + 10^3 = 1^3 + 12^3 = 1729)$$

Due to poor health, Ramanujan died at the early age of 32, one year after returning to India to be with his wife whom he left at the age of 25. His ***Lost Notebook***, found in 1976, contains about 4,000 formulas and other work. Ramanujan's formulas continue to keep mathematicians busy today, waiting to be deciphered and proved - many of which were given in the final form without any intermediate steps.