Bell Numbers and Bell Numbers Modulo a Prime Number

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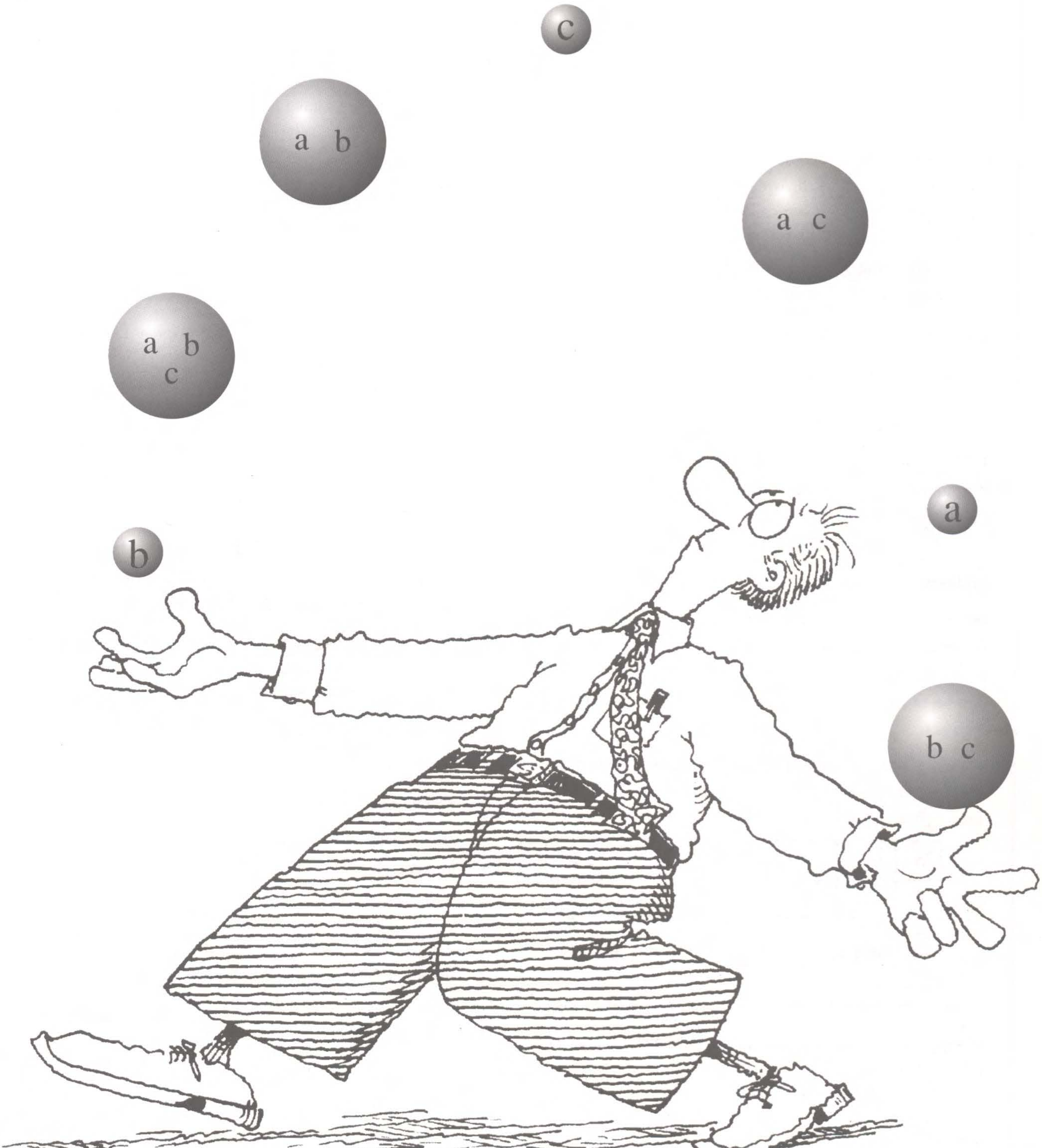

1. What are the Bell numbers?

How many ways can we split the set $\{a, b, c\}$ into smaller sets? For example, we can split it into two sets $\{a, b\}$, $\{c\}$. We call this a partition of the set $\{a, b, c\}$. We consider $\{a, b, c\}$ itself a partition of $\{a, b, c\}$ too. The complete list of partitions of $\{a, b, c\}$ is:

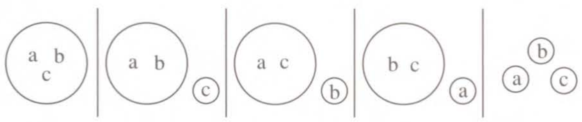

Figure 1: The partitions of $\{a,b,c\}$.

So there are 5 ways to form partitions of $\{a, b, c\}$. What about the set $\{a, b, c, d\}$? The answer is 15. (Try writing out all the 15 possible ways.) If we denote the number of ways of splitting a set with n elements by B_n, we know that $B_1 = 1$, $B_2 = 2$ and $B_3 = 5$. These numbers are called the *Bell numbers*. They are named after E. T. Bell (1883–1960), an American mathematician who wrote about them in 1934 [1]. But they have appeared much earlier in 1907 in works of S. Minetola [6].

Other than counting the number of ways to split a set, they also count the number of factorizations of certain numbers. For example, how many ways can the number 30 be factorized?

$$30 = 2 \times 15 = 3 \times 10 = 5 \times 6 = 2 \times 3 \times 5.$$

Including the number 30, the answer is 5 which is B_3. In general, if a number N is a product of n different prime numbers, then the number of factorizations of N is B_n. In our previous example, $30 = 2 \times 3 \times 5$ and 2, 3, 5 are all prime numbers.

Problem

Prove that if a number N is a product of n different prime numbers, then the number of factorizations of N is B_n. (Hint: Find a bijection between the factorizations of N and the partitions of a set of size n. Invoke the Bijection Principle in [4] to get the answer.)

Here is a table of the Bell numbers. Note that B_0 is given the value 1.

n	0	1	2	3	4	5
B_n	1	1	2	5	15	52

Table 1: Bell numbers (up to $n = 5$).

A list of the Bell numbers up to B_{74} can be found in [5].

We do not have to list down all the partitions of a set of size 6 to find B_6.

There is a formula that can help us calculate B_{n+1} using the values of $B_0, B_1, \ldots, B_n$. The formula is

$$B_{n+1} = B_n + nB_{n-1} + \binom{n}{2}B_{n-2} + \ldots + \binom{n}{n-1}B_1 + B_0$$
$$= \sum_{i=0}^{n} \binom{n}{i} B_{n-i}. \qquad (1)$$

(The numbers $\binom{n}{k}$ are called binomial coefficients. See [3] for more details.) For example, to calculate B_6, we substitute the values in Table 1 into (1). So

$$B_6 = B_5 + 5B_4 + 10B_3 + 10B_2 + 5B_1 + B_0$$
$$= 52 + 5 \times 15 + 10 \times 5 + 10 \times 2 + 5 \times 1 + 1$$
$$= 203.$$

We can use the techniques described in [2] to prove the formula (1). Consider the set $\{1, 2, \ldots, n, n+1\}$ which has $n+1$ elements. First, we will use the Addition Principle.

To count all the possible partitions of $\{1, 2, \ldots, n, n+1\}$ we divide into cases according to the order of the subset containing 1 in the partition. Then in each case, we will use the Multiplication Principle.

Case 1: The subset containing 1 has order 1.
In this case, $\{1\}$ forms part of the partition. We just have to partition the set $\{2, 3, \ldots, n+1\}$ and there are B_n ways to do this.

Case 2: The subset containing 1 has order 2.
We need another number to form the subset containing 1. This number has to be chosen from $\{2, 3, \ldots, n+1\}$. So there are n ways to do this. Next, we are left with $n-1$ numbers to form the rest of the partition and there are B_{n-1} ways to do so. Hence the total number of ways in this case is nB_{n-1}.

Do you see the reasoning? We repeat the argument for the kth case.

Case *k*: The subset containing 1 has order k.

To form the subset containing 1, we need to choose $k-1$ other elements from $\{2, 3, \ldots, n+1\}$. So there are $\binom{n}{k-1}$ ways. After choosing these $k-1$ numbers, we have $n-k+1$ numbers left to complete the partition. That is, we are forming a partition out of $n-k+1$ numbers. There are B_{n-k+1} ways to do this. By the Multiplication Principle, there are $\binom{n}{k-1}B_{n-k+1}$ ways to form partitions such that the subset containing 1 has order k.

Adding up all the possible cases gives

$$B_{n+1} = \binom{n}{0}B_n + \binom{n}{1}B_{n-1} + \ldots + \binom{n}{n-1}B_1 + \binom{n}{n}B_0$$

$$= \sum_{i=0}^{n}\binom{n}{i}B_{n-i}.$$

2. Modulo a Prime Number

The Bell numbers B_n grow very big as n increases. For example, $B_{10} = 115975$ and B_{20} has 14 digits. One way to deal with this is to look at the Bell numbers modulo some number, in particular a prime number. That is, we find the remainder when each is divided by a prime number.

n	B_n	(mod 2)	(mod 3)	(mod 5)
0	1	1	1	1
1	1	1	1	1
2	2	0	2	2
3	5	1	2	0
4	15	1	0	0
5	52	0	1	2
6	203	1	2	3
7	877	1	1	2
8	4140	0	0	0
9	21147	1	0	2
10	115975	1	1	0

Table 2: Bell numbers modulo 2, 3 and 5

(A table of B_n modulo p up to $p = 47$ can be found in [5].) It turns out that the remainders of Bell numbers after dividing by a prime number satisfy some nice properties. When $p = 2$, the sequence of Bell numbers become 1, 1, 0, 1, 1, 0, ...

It is just 1, 1, 0 repeating itself. This pattern is called the *least residue pattern* and we say that the Bell numbers modulo 2 is *periodic* with *period* 3. The least residue pattern is the shortest sequence that repeats itself and the period is the length of the least residue pattern. From this, we can tell that the sequence of Bell numbers come in the form of two odd numbers followed by one even number and then two odd numbers again and so on.

When $p = 3$, the period is 13 and the least residue pattern is 1, 1, 2, 2, 0, 1, 2, 1, 0, 0, 1, 0, 1. When $p = 5$, the period is 781. In general, it has been proved [10] that the Bell numbers modulo a prime number p is periodic and the period must divide $\frac{p^p - 1}{p - 1}$. Let us denote this by N_p. That is to say

$$B_{n+N_p} \equiv B_n \pmod{p} \qquad (2)$$

for all $n = 0, 1, \ldots$. It has been checked for all primes up to $p = 101$ and some others that this is indeed the period [9]. But it is still not known if this is true for all primes.

One way to check the period is to generate the Bell numbers modulo a prime number p. We do not need to know the value of B_n to calculate $B_n \pmod{p}$. There is a simple formula for doing this:

$$B_{n+p} \equiv B_n + B_{n+1} \pmod{p} \qquad (3)$$

which was proved by Touchard [8]. (Note: This article is written in French. For an English version, see [7] or [10]). For example, to compute B_n (mod 5), we use the first 5 values:

$$B_0 \equiv 1,\ B_1 \equiv 1,\ B_2 \equiv 2,\ B_3 \equiv 0,\ B_4 \equiv 0 \pmod{5}$$

Then applying formula (3),

$$B_5 \equiv B_0 + B_1 \equiv 2 \pmod{5}$$

$$B_6 \equiv B_1 + B_2 \equiv 3 \pmod{5}$$

$$B_7 \equiv B_2 + B_3 \equiv 2 \pmod{5}$$

$$B_8 \equiv B_3 + B_4 \equiv 0 \pmod{5}$$

and so on. We will not present a proof of (3) as the techniques are beyond this article.

There are other uses for the formula (3). When we substitute $n = 0$ into (3), we get

$$B_p \equiv B_0 + B_1 \equiv 2 \pmod{p} \qquad (4)$$

So no matter which prime number p is, as long as $p \neq 2$, B_p always leaves a remainder 2 when divided by p. In the case when $p = 2$, B_2 is even and so the remainder is 0.

Another use of (3) is in solving the following problem.

Problem

Show that the sum of N_p consecutive Bell numbers is divisible by p.

Solution

Since the period of the Bell numbers modulo a prime divides N_p, the sum of any N_p consecutive Bell numbers must have the same remainder. For example, let $S = B_0 + B_1 + \ldots + B_{N_p-1}$ and $S' = B_1 + B_2 + \ldots + B_{N_p}$. From (2), $B_{N_p} \equiv B_0 \pmod{p}$.

That means

$$S' = B_1 + B_2 + \ldots + B_{N_p-1} + B_{N_p}$$

$$\equiv B_1 + B_2 + \ldots + B_{N_p-1} + B_0$$

$$\equiv B_0 + B_1 + \ldots + B_{N_p-1}$$

$$\equiv S \pmod{p}.$$

Similarly, if we let $S'' = B_p + B_{p+1} + \ldots + B_{N_p+p-1}$, then $S'' \equiv S \pmod p$. This means that we just need to find the remainder when S is divided by p.

From formula (3),

$$B_0 + B_1 \equiv B_p \pmod p$$

$$B_1 + B_2 \equiv B_{p+1} \pmod p$$

$$\ldots$$

$$B_{N_p-1} + B_{N_p} \equiv B_{N_p+p-1} \pmod p.$$

The terms on the righthand side add up to S''. The first terms on the lefthand side of each equation add up to S and the second terms add up to S'. So if we add up all these equations, we get

$$S + S' \equiv S'' \pmod p$$

$$\Rightarrow S + S \equiv S \pmod p$$

$$\Rightarrow S \equiv 0 \pmod p.$$

So, the remainder is zero which means that the sum of any N_p consecutive Bell numbers is divisible by p.

We hope that this article would spark some interest in the Bell numbers. There are many other identities such as (3) and (4). We challenge the reader to find some of them. M^2

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* *This article is based on an Honours Thesis by Teo T. K. [7].*