Sketching the cubic curves

by

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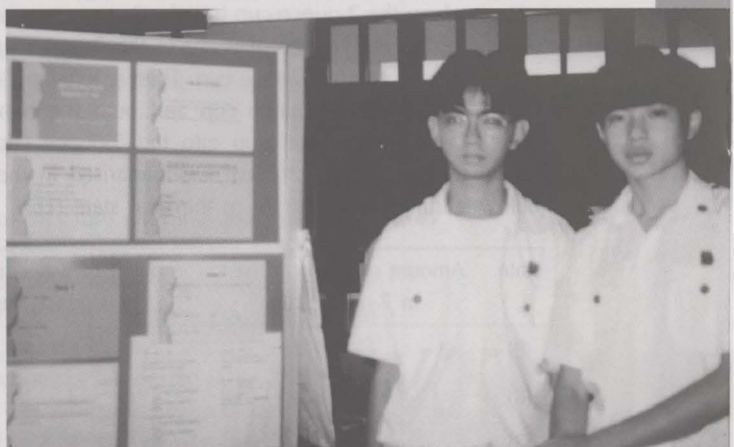


The project on display with two of the authors Warren Phee (left) and James Fu (right).

* *The authors were secondary four students from The Chinese High School and the article is modified from their project with the same title which won the gold medal in the Upper Mathematics Section of the School's Projects Day held on 20 July 1996.*

Objectives

It is often a tedious job to plot a cubic curve, $y = ax^3 + bx^2 + cx + d$. Usually, we differentiate the equation and set dy/dx = 0 to find the x -coordinate of the stationary points. But what if dy/dx $\neq$ 0 for all real values of x? What would be the features of the corresponding curves? What values of the parameters a, b, c and d would cause such curves to occur?

Notice that the shape of the cubic curve can vary. Cubic curves are interesting as they can have two, one or no stationary points. These simple observations led us to start on this project of finding an alternative method for sketching cubic curves.

Procedure

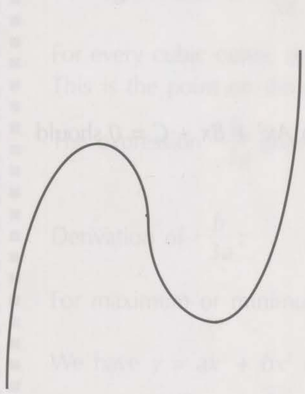
To determine:

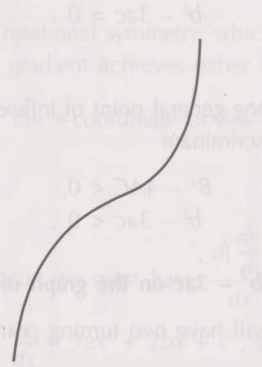
1. the shape of the curve, we check whether $a > 0$ or $a < 0$.
2. where the curve cuts the y-axis, we check whether $d > 0$ or $d < 0$.
3. whether the curve has 2 stationary points, 1 stationary point of inflexion or 1 general point of inflexion, we check whether $b^2 - 3ac > 0$, $b^2 - 3ac = 0$ or $b^2 - 3ac < 0$.
4. the x-coordinate of the point of rotational symmetry, we check the value of $-b / 3a$.

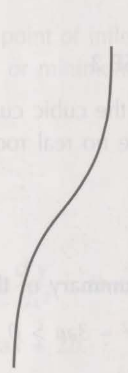
The significance of a

For $a > 0$, when $x \to +\infty$, $ax^3 \to +\infty$ $\quad \therefore\ y \to +\infty$
when $x \to -\infty$, $ax^3 \to -\infty$ $\quad \therefore\ y \to -\infty$.

The shape of the cubic curve would be one of the following:

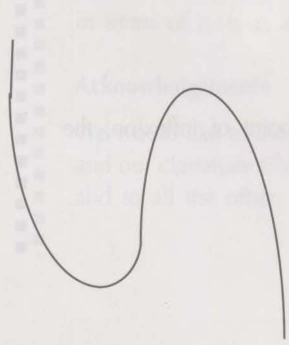

curve with 2 turning points

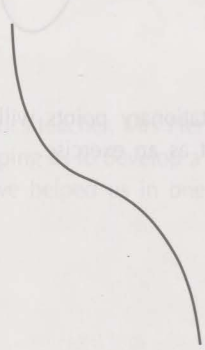

curve with 1 stationary point of inflexion

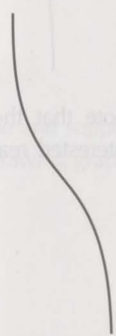

curve with 1 general point of inflexion

For $a < 0$, when $x \to +\infty$, $ax^3 \to -\infty$ $\quad \therefore\ y \to -\infty$
when $x \to -\infty$, $ax^3 \to +\infty$ $\quad \therefore\ y \to +\infty$.

The shape of the cubic curve would be one of the following:


curve with 2 turning points


curve with 1 stationary point of inflexion


curve with 1 general point of inflexion

The significance of d

"d" is the y-coordinate of the point at which the curve cuts the y-axis.

The significance of $b^2 - 3ac$

This expression determines whether the cubic curve has two stationary points, one stationary point of inflexion or one general point of inflexion.

Derivation of $b^2 - 3ac$: we have $y = ax^3 + bx^2 + cx + d$, so $\frac{dy}{dx} = 3ax^2 + 2bx + c.$

Let $\frac{dy}{dx} = 0.$ We have $3ax^2 + 2bx + c = 0.$ Set $3ax^2 + 2bx + c = Ax^2 + Bx + C,$

where $A = 3a$, $B = 2b$ & $C = c$.

CASE 1

For the cubic curve to have two stationary points, the equation $Ax^2 + Bx + C = 0$ should have two real and distinct roots. $\therefore$ the discriminant

$$B^2 - 4AC > 0,$$
$$(2b)^2 - 4(3a)(c) > 0,$$
$$4(b^2 - 3ac) > 0,$$
$$b^2 - 3ac > 0.$$

CASE 2

For the cubic curve to have one stationary point of inflexion, the euqation $Ax^2 + Bx + C = 0$ should have two real and identical roots. $\therefore$ the discriminant

$$B^2 - 4AC = 0,$$
$$b^2 - 3ac = 0.$$

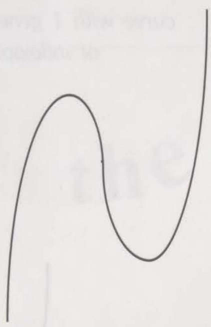
CASE 3

For the cubic curve to have one general point of inflexion, the equation $Ax^2 + Bx + C = 0$ should have no real roots. $\therefore$ the discriminant

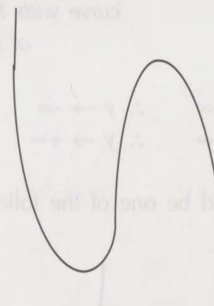
$$B^2 - 4AC < 0,$$
$$b^2 - 3ac < 0.$$

A summary of the effect of $b^2 - 3ac$ on the graph of a cubic curve

If $b^2 - 3ac > 0$, the curve will have two turning points.

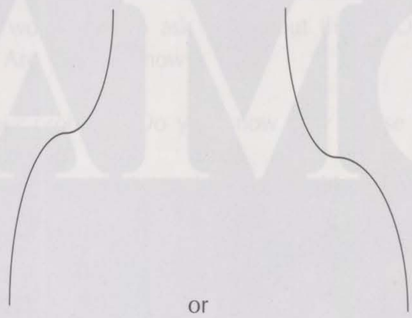


or



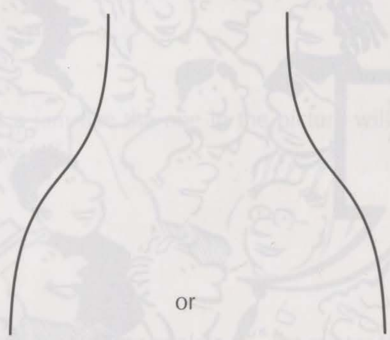
Note that the mid-point of the two stationary points will be a general point of inflexion; the interested reader should derive this fact as an exercise.

If $b^2 - 3ac = 0$, the curve will have a stationary point of inflexion.



or

If $b^2 - 3ac < 0$, the curve will have a general point of inflexion.



or

The significance of $-\dfrac{b}{3a}$

For every cubic curve, there is a point of rotational symmetry, which is also the point of inflexion. This is the point on the curve where the gradient achieves either its maximum or minimum.

The expression $-\dfrac{b}{3a}$ gives us the value of the x-coordinate of this point.

Derivation of $-\dfrac{b}{3a}$:

For maximum or minimum gradient to occur, we must have $\dfrac{d[\frac{dy}{dx}]}{dx} = 0$, i.e. $\dfrac{d^2y}{dx^2} = 0$.

We have $y = ax^3 + bx^2 + cx + d$. Thus $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$, and $\dfrac{d^2y}{dx^2} = 6ax + 2b$.

Now $6ax + 2b = 0$, so $6ax = -2b$, $x = -\dfrac{b}{3a}$.

Areas for further study

In the course of our project, we did not manage to determine the number of times the curve cuts the x-axis and thus were not able to sketch an even more accurate curve. We hope that in the future, we will find a way to determine how many times and where the cubic curve cuts the x-axis in terms of a, b, c, and d.

Acknowledgements

We would like to thank our Mathematics teacher, Mrs Heng, for her invaluable advice and support, and our classmate Chong Jiayi, for helping us to develop a windows programme for drawing graphs, and to all the other people who have helped us in one way or another. M^2