what is APPLICABLE MATHEMATICS?

by Lee Peng Yee

Pure, applied, and applicable mathematics

We study arithmetic in primary schools, and later algebra, geometry, trigonometry and others in secondary schools. At the university, we study calculus and algebra, or sometimes we prefer to call it analysis and algebra, or pure mathematics. Also, we study mechanics and other related subjects, which we call applied mathematics; and we study statistics. About twenty or thirty years ago, if students wanted to study statistics, they had to complete their undergraduate studies first and then take a postgraduate diploma course in statistics. Now our students do it at the undergraduate level and even in schools.

Recently two other subjects have also been taught at the university level, namely, numerical analysis and operational research. In fact, the first department of operational research in a British university was established in 1965. Of course, there are many such departments now in universities all over the world. These subjects, namely, statistics, numerical analysis, and operational research, are applications of mathematics. In order to distinguish them from classical applied mathematics, we call them applicable mathematics.

Both in applied mathematics and in applicable mathematics, we try to apply mathematics to solve problems. However the problems which we try to solve may be different, so are the methods used; and above all the flavour of applicable mathematics is very different from that of applied mathematics. We shall give a few examples to illustrate our points.

Mathematics is abstract and mathematics is useful

In order that mathematics can become useful, it has to be made abstract. For example, the number "three" is an abstract concept, it is not three oranges or three apples, but the common property that they have which we call "three". It is necessary to make it abstract so that we can do arithmetic and then apply it to different situations which are not necessarily restricted to oranges or apples.

To solve a problem in the real world using mathematics, first of all we have to convert the real problem into a mathematical problem. Then we solve the mathematical problem, if possible, mathematically. However it might not always be possible to solve a problem. Even if we have solved the mathematical problem, it does not mean that we have solved the problem in the real world, because we still have to interpret the mathematical solution. And sometimes the mathematical solution does not have a meaning in the real world.

For example, suppose we want to measure the area of a rectangular table top. So we measure the length and the width, and draw a rectangle using the given length and width. By doing so, we have converted the problem into finding the area of a rectangle, which is a mathematical problem. The product of the length by the width gives an approximation of the area of the table top. Hence the problem is solved, though only approximately. Basically, the procedure is as follows:

$$\boxed{\text{Real problem}} \rightarrow \boxed{\text{Mathematical model}} \rightarrow \boxed{\text{Mathematical solution}} \rightarrow \boxed{\text{Interpretation}}$$

To solve a mathematical problem, very often we set up an equation and try to solve it. The equation may have a solution or it may not, or the solution may be *complex* (involving $\sqrt{-1}$) which does not have a meaning in the real world. So we may have solved the equation without actually solving the given problem.

It is easy to imagine how to construct a mathematical model for an engineering problem. However it is not so easy to imagine that for an economic problem. A very different approach is required. The idea is to describe quantitatively the problems which are not very quantitative in nature. This is the flavour and essence of applicable mathematics.

Exact, approximate and probabilistic methods

There are problems which have exact solutions. There are problems which have exact solutions, but the exact solutions are not easy to find and therefore we look for some approximate solutions. There are problems which do not have exact solutions, but they have approximate solutions. However there are even problems which have neither exact nor approximate solutions, yet they still have some kind of solution, perhaps a probabilistic solution.

Let me give an example of the last case. Suppose that we went to Kuala Lumpur and saw some fish that we would like to bring back to Singapore. Certainly we would like to have both male and female if possible. Suppose that it is not easy to distinguish the male from the female since the fish are very small. The question is: What is the minimum number of the fish which we should bring back? The answer is at least two. The best situation is one male and one female. But it may happen that both are male or both are female. Of course if we brought plenty home, the chances of having both male and female are greater. Could we bring less, but how much less? This problem does not have an exact answer and it does not have an approximate answer either. However it has a probabilistic answer in the form that if you bring a certain number of fish back then the chances of having both a male and a female can be calculated precisely. Even though the answer is a probabilistic one, it is still useful.

In order to face all these different kinds of problems, we need different kinds of methods to solve them. Some are exact, some approximate, and some even probabilistic. Let me give one example. Suppose we want to find the area of a circle. Obviously, we may use the formula $A = \pi r^2$ which is an exact method. Alternatively, we may draw equally spaced horizontal and vertical lines and count the number of squares inside the circle. This is an approximate method. The closer the lines, the more accurate approximation we obtain.

We may also employ a probabilistic method. Draw the circle on a rectangular sheet, and hang it on the wall. Throw darts at it from a distance. It is a hit if the dart lands inside the circle, otherwise it is a miss. Count the number of hits and calculate the ratio of the number of hits to the total number of throws. In this way we obtain the ratio of the area of the circle to that of the rectangle. It is easy to compute the area of the rectangle, and we can now compute the area of the circle. We can carry out the experiment many times if necessary, and we should obtain reasonably good result. What is interesting is that the exact method works only with regular figures whose areas can be computed by known formulae whereas the approximate and the probabilistic methods work with any shape of figures. So the other two methods, though nonexact, have their merit. Very often in applicable mathematics we are concerned with nonexact methods rather than exact methods. As we can see, approximate and probabilistic methods are powerful and they can work in situations where exact methods fail.

Two examples in statistics

Suppose that we want to calculate the population of a certain species of bird. One way is to catch them or shoot them all, and count them. Of course, this is not a practical method. However there is a statistical way of calculating the population. In fact, this is also the way currently used by naturalists.

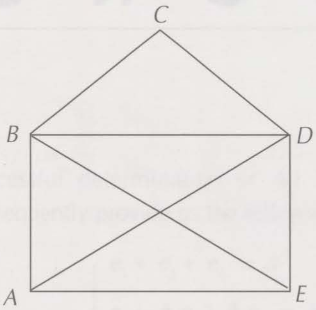
Imagine that we want to calculate the number of white beans in a big bag. One way is to count them. Another is to get hold of a small bag of red beans, pour it into the big bag, and mix it thoroughly. Then scoop out a handful of beans containing both red and white. Count them. Again take the ratio. If we know beforehand the number of red beans, using the ratio we can calculate approximately the number of white beans. Since there are fewer red beans than white beans, the job is easier. Further we may experiment several times if necessary. In terms of the bird population, we catch some of them first, tag them, and release them again. After a period, we catch another batch of the species and count them again. By regarding the tagged ones as red beans and untagged ones white beans, we obtain approximately the population of the species. Of course, there are many refinements to the method. What we have explained above is the basic principle.

Let me give another example involving coding and decoding. Take a passage from the newspaper, and interchange some of the letters of the alphabet. So it becomes a coded message. Can we decode it? Certainly we can. All we need to do is to try all the possibilities. Is there another way? Let us apply statistics.

First, collect some pages of the newspaper and do a survey of the frequency of the appearance of each letter of the alphabet in the paper. Compare with that of the coded passage. Try to match them as much as we can. If we can decode a fair proportion of the coded letters, then we can deduce the rest. Finally, the message is decoded. The actual situation may be more complicated than this. What we have described here is just a general idea. Here we have given examples to show how some problems may be solved using different mathematical methods.

An example in operational research

There is a famous problem in operational research called the Chinese Postman Problem. Suppose that in a small town there is a postman who delivers mail to each household on foot. If he starts from the post office, does one round and comes back to the post office again, can he design a route so that he walks the minimum amount of distance? If possible, he should walk each street once only. First, let us convert it into a mathematical problem. Consider the streets as lines and the junctions as points, then we can draw a diagram and call it a network or a graph. Then the question is to design the shortest route passing through each street at least once, starting from a point and coming back to the same point again. This is a problem in a branch of mathematics called graph theory.



Consider the following simple example.

Suppose we start from A. We can go $A \to B \to C \to D \to E \to B \to D \to A \to E \to A$. Unfortunately, we have to pass through the street or the line AE twice. We may try some other ways. Let us look at the diagram more carefully. We find that B, C and D have an even number of lines connected to them whereas A and E have an odd number of lines connected to them. If we start from A, go out on one line, come back on another, and go out again on the third line, that exhausts all the lines. To come back to A again, we have no choice but to repeat one of the lines. In other words, to start from A and come back to A again without repeating the lines, A must have even number of lines connected to it. Since A has only odd number of lines connected to it, the above scheme is one of the best possible. Hence we have solved the mathematical problem. Hopefully, we may apply this to the real situation.

An example in numerical analysis

We can solve a quadratic equation completely. With some effort, we can also solve a cubic equation. However it becomes harder and harder if not impossible to solve a polynomial equation of higher degree or an equation involving sine and cosine. If we are allowed to use a different approach such as numerical methods, then the problem may become simple. We shall give one example.

Solve the equation

$$x = \cos x.$$

We can draw the graph of $y = x$ and that of $y = \cos x$. The intersection point is the solution of the equation $x = \cos x$. Alternatively, let $x_1 = 1$. Then compute $x_2 = \cos x_1 = 0.5403023059$, and subsequently,

$$x_3 = \cos x_2 = 0.8575532158,$$

$$x_4 = \cos x_3 = 0.6542897905 \text{ and so on.}$$

If one is using a calculator with 10 digits of accuracy, eventually, the number repeats itself. Hence the approximate solution is 0.7390851332. This is called the iterative method. We take an initial approximation x_1. Then apply the iteration

$$x_{n+1} = \cos x_n \quad \text{for } n = 1, 2, \ldots .$$

When the value repeats itself on the calculator, that value is the answer. Though it does not always work, we can find out conditions under which the iteration method works. It so happens that the above example works. This is a very powerful and indeed typical method in numerical analysis.

In an iteration, the first question we ask is whether the iteration converges, that is, whether the iteration gives us the required answer. Secondly, we are interested in the accuracy of the approximation. Thirdly, if the iteration converges and the accuracy is good, then we want to know how fast the iteration converges, that is, how many times we have to perform the iteration before we arrive at the required result. With the advances in computer technology, numerical analysis is a fast growing subject in mathematics.

Existence versus algorithm

There is a distinctly different flavour in applicable mathematics. In a way, applicable mathematics can be classified as part of modern mathematics. In classical mathematics, when we have a problem, we ask first of all whether there is a solution, that is, whether a solution exists. Then we want to know precisely what the answer is. Very often we express the answer in some definite form. In applicable mathematics, we are interested in feasible solutions; in other words, not exactly the answer we want but something close to it. Then we construct an algorithm or a procedure through which we get close to the solution step by step. We can see this from the few examples we have given above. Though applicable mathematics is not taught in schools except for some statistics, it has already become a standard part of the undergraduate mathematics course at the university level. M²

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