Problems Corner

Contest

Prizes in the form of book vouchers will be awarded to the first received best solution submitted by secondary school or junior college students in Singapore for each of these problems.

To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

Solutions should be typed and sent to:

The Editor
Mathematical Medley
c/o Department of Mathematics
National University of Singapore
Kent Ridge, Singapore 119260

and should arrive before 31 December 1996.

The Editor's decision will be final and no correspondence will be entertained.

Problem 3

Prize

One $100 book voucher

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be four polynomials in x which satisfy the identity

$$P(x^4) + xQ(x^4) + x^2R(x^4) \equiv (x^3 + x^2 + x + 1)S(x).$$

Prove that $(x - 1)$ is a common factor of $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Problem 4

Prove that

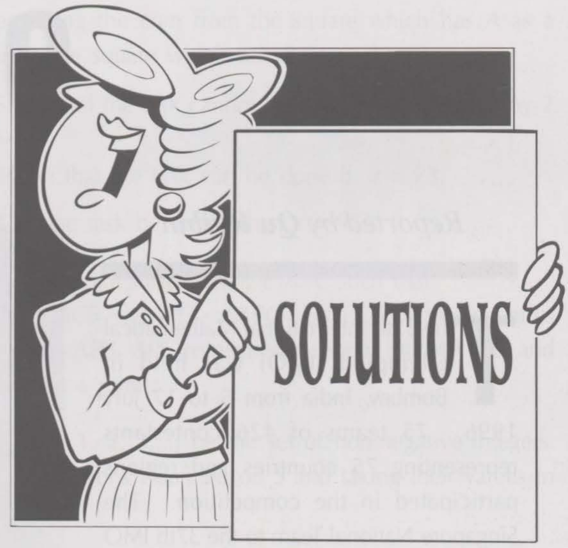
$$1 + \frac{1}{1996}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1996}\right) > (1997)^{\frac{1}{1996}}$$

Prize

One $100 book voucher

Solutions

Solutions to the problems in Volume 23 No. 1 March 1996



Solution to Problem 1

by G. Venkateswara Rao, Hwa Chong JC, Class 96S32.

We have

$$1996 \cdot 1994 \cdot 1992 \cdots 2 \equiv (-1)(-3)(-5) \cdots (-1995) \pmod{1997}$$

$$\therefore \quad \overline{1996} \equiv (-1)^{998}\ \overline{1995} \pmod{1997}$$

$$\equiv \overline{1995} \pmod{1997}$$

$$\therefore \overline{1996} - \overline{1995} \text{ is divisible by } 1997.$$

Solved also by Eric Fang Kin Meng, RJC, Pang Chin How Jeffrey, Hwa Chong JC, Class 96S32. Three incorrect solutions were received.

Editor's Note: The editors have decided to increase the prize money for this problem to \$90 to be shared equally by G. Venkateswara Rao, Eric Fang Kin Meng and Pang Chin How Jeffrey.

Solution to Problem 2

by the Editors.

Let u_n denote the number of ways to group the $2n$ cards into n pairs such that the two numbers in each pair are either equal or differ by 1.

For $n = 1$ or $n = 2$, any pairing will satisfy the required condition and so we have $u_1 = 1$ and $u_2 = 3$. For $n > 2$, consider one of the two cards numbered n. If we group this with the other card numbered n, then there are u_{n-1} ways to group the remaining $2n - 2$ cards. Otherwise we must group this with either one of the two cards numbered $n - 1$ (note that there are two ways to do this) and the other card numbered n must be grouped with the remaining one numbered $n - 1$ and then there are u_{n-2} ways to group the remaining $2n - 4$ cards. Therefore, we have

$$u_n = u_{n-1} + 2u_{n-2}.$$

The characteristic equation of this recurrence relation is $x^2 - x - 2 = 0$. Therefore $x = -1$ or 2. Therefore $u_n = a2^n + b(-1)^n$ where a and b are constants.

From $u_1 = 1$ and $u_2 = 3$, we have $a = \frac{2}{3}$ and $b = \frac{1}{3}$. Therefore

$$u_n = \frac{2^{n+1} + (-1)^n}{3}$$

Editor's note: No correct solution was received. The editors have decided to take \$40 from the prize money for Problem 2 and add this to the prize money for Problem 1 to bring the total amount of the latter to \$90.