On Menelaus' Theorem

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In our preceding article [1], we introduced the celebrated Ceva's Theorem and its converse which is stated as follows:

> The cevians AP, BQ and CR of ΔABC are concurrent if and only if
>
> $$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1.$$

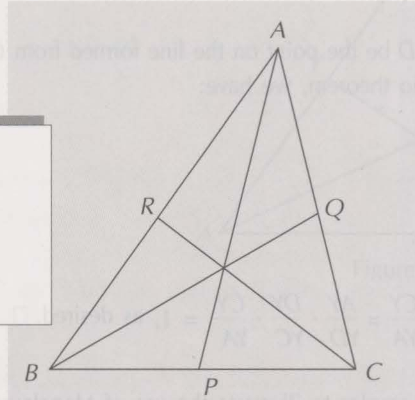

Figure 1

Three distinct points on a plane are said to be *collinear* if they lie on a straight line. Given ΔABC, let X, Y and Z be, respectively, points other than the vertices A, B, C, on the lines formed from sides BC, CA and AB as shown in Figure 2. Ceva's theorem and its converse provide us with a criterion to determine whether three given cevians are concurrent. We may ask: is there a criterion which will enable us to determine whether the three given points as shown in Figure 2 are collinear?

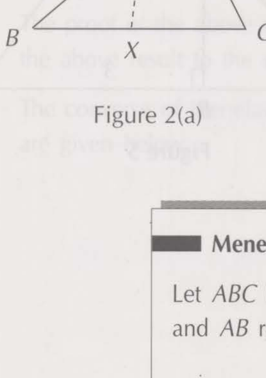

Figure 2(a)

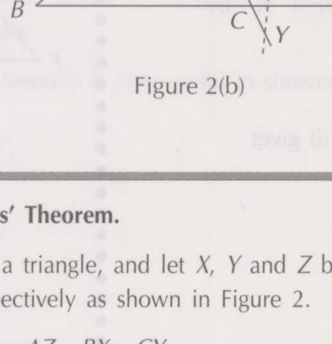

Figure 2(b)

While Ceva's theorem was established in the 17th century, a positive answer to the above question was given two thousand years ago by Menelaus of Alexandria (about 98A.D.). In this article, we shall introduce this important result and also show some of its applications.

Menelaus' Theorem.

Let ABC be a triangle, and let X, Y and Z be points on the lines formed from BC, CA and AB respectively as shown in Figure 2. If X, Y and Z are collinear, then

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1. \qquad (1)$$

There are several different proofs of Menelaus' theorem. In what follows, we give two of them; the first proof applies the notion of area, and the second proof uses the ratio theorem.

First Proof

We denote by (PQR) the area of ΔPQR.

Consider Figure 3. As was shown in [1], we have

$$\frac{AZ}{ZB} = \frac{(AYZ)}{(BYZ)}$$

$$\frac{BX}{XC} = \frac{(BYZ)}{(CYZ)}$$

and $$\frac{CY}{YA} = \frac{(CYZ)}{(AYZ)}.$$

Thus $$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{(AYZ)}{(BYZ)} \cdot \frac{(BYZ)}{(CYZ)} \cdot \frac{(CYZ)}{(AYZ)} = 1,$$ as required.

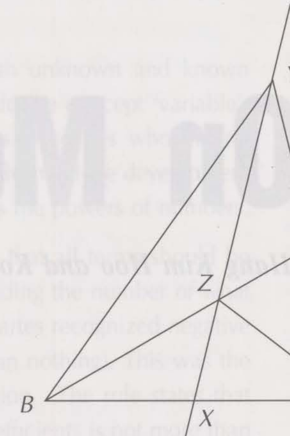

Figure 3

Second Proof

As shown in Figure 4, let D be the point on the line formed from CA such that $BD // XY$. Then by the ratio theorem, we have:

$$\frac{AZ}{ZB} = \frac{AY}{YD}$$

and $$\frac{BX}{XC} = \frac{DY}{YC}$$

Thus $$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{AY}{YD} \cdot \frac{DY}{YC} \cdot \frac{CY}{YA} = 1,$$ as desired. □

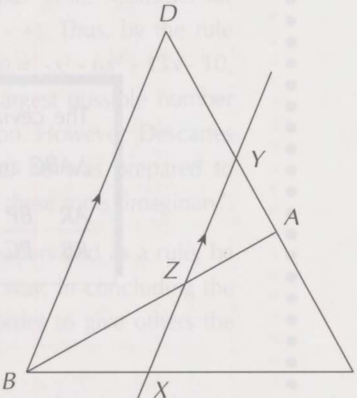

Figure 4

We shall now give two examples to illustrate the use of Menelaus' theorem.

Example 1

In Figure 5, ABC is a triangle with $\angle B = 90^\circ$, $BC = 3cm$ and $AB = 4cm$. D is a point on AC such that $AD = 1cm$, and E is the mid-point of AB. Join D and E, and extend DE to meet CB extended at F. Find BF.

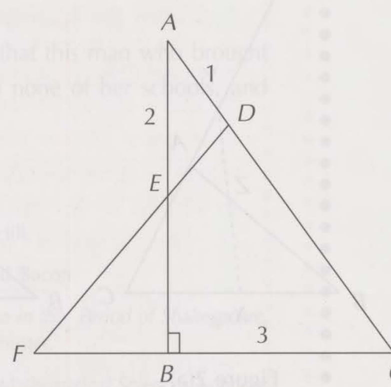

Figure 5

Solution

Consider ΔABC. Then D, E and F are, respectively, points on the sides CA, AB and BC, and by construction are collinear. By Menelaus' theorem,

$$\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CD}{DA} = 1. \qquad \text{(i)}$$

By assumption, $AE = EB = 2$, $DA = 1$ and $FC = FB + BC = BF + 3$. By Pythagoras' theorem,

$$AC = \sqrt{BC^2 + AB^2} = \sqrt{3^2 + 4^2} = 5,$$

and so $CD = AC - AD = 5 - 1 = 4$. Substituting these data into (i) gives

$$\frac{2}{2} \cdot \frac{BF}{BF + 3} \cdot \frac{4}{1} = 1.$$

Solving for BF yields $BF = 1$. □

In applying Menelaus' theorem, we need to identify a trianlge and three collinear points respectively on its sides. (Thus, in Example 1, we take ΔABC and the points D, E and F.) To simplify notation, in what follows, in Menelaus' theorem we refer to the lines YZX in Figure 2(a) and ZXY in Figure 2(b) as the *transversals* of ΔABC.

Example 2

In Figure 6, ABC is a triangle, X and Y are points on BC and CA respectively, and R is the point of intersection of AX and BY.

Given $\frac{AY}{YC} = P$ and $\frac{AR}{RX} = q$, where $0 < p < q$, express $\frac{BX}{XC}$ in terms of p and q.

Solution

Consider ΔAXC and its transversal BRY. By Menelaus' theorem,

$$\frac{AR}{RX} \cdot \frac{XB}{BC} \cdot \frac{CY}{YA} = 1.$$

Thus $$\frac{BC}{XB} = \frac{AR}{RX} \cdot \frac{CY}{YA} = \frac{q}{p},$$

i.e., $$\frac{BX + XC}{BX} = \frac{q}{p}.$$

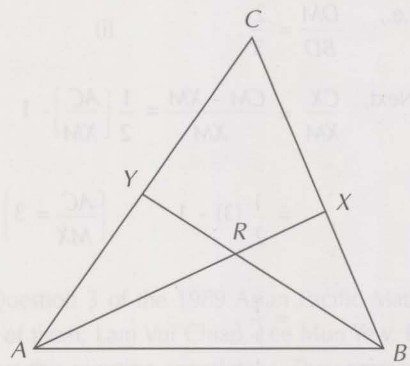

Figure 6

It follows that

$$1 + \frac{XC}{BX} = \frac{q}{p},$$

$$\frac{XC}{BX} = \frac{q}{p} - 1 = \frac{q - p}{p},$$

i.e., $$\frac{BX}{XC} = \frac{p}{q - p}. \quad \square$$

Let X, Y and Z be, respectively, points on the sides BC, CA and AB of ΔABC as shown in Figure 2. Menelaus' theorem states that if X, Y and Z are collinear, then equality (1) holds. Does the converse of Menelaus' theorem also hold? That is, if X, Y and Z are points such that equality (1) holds, are they always collinear? A positive answer to this question is given in the following result.

The Converse of Menelaus' Theorem.

Let X, Y and Z be points on the lines formed from the sides BC, CA and AB of ΔABC respectively.

If $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1$, then X, Y and Z are collinear.

The proof of the above result is similar to the proof of the converse of Ceva's theorem as given in [1]. We leave the proof of the above result to the reader.

The converse of Menelaus' theorem is very useful in showing the collinearity of three given points on a plane. Two examples are given below.

Example 3

In Figure 7, the diagonals AC and BD of a quadrilateral $ABCD$ meet at M in such a way that $AM = MC$ and $DM = 2MB$. Suppose that X and Y are points on MC and BC respectively such that

$$\frac{AC}{MX} = \frac{BY}{YC} = 3.$$

Show that the points D, X and Y are collinear.

Proof

First, we have $\frac{DM}{BD} = \frac{DM}{BM + MD} = \frac{2MB}{3MB}$ $(DM = 2MB)$

$$= \frac{2}{3},$$

i.e., $\frac{DM}{BD} = \frac{2}{3}$. (i)

Next, $\frac{CX}{XM} = \frac{CM - XM}{XM} = \frac{1}{2}\left[\frac{AC}{XM}\right] - 1$ $(AM = MC)$

$$= \frac{1}{2}(3) - 1 \qquad \left[\frac{AC}{MX} = 3\right]$$

$$= \frac{1}{2},$$

i.e., $\frac{CX}{XM} = \frac{1}{2}$ (ii)

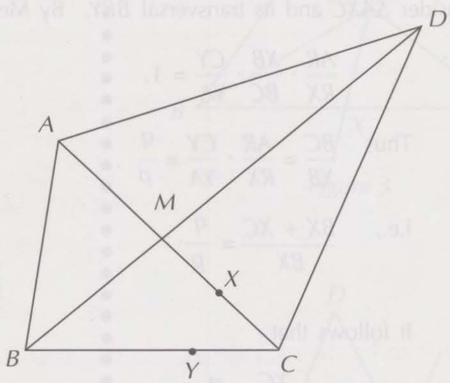

Figure 7

Now, consider ΔMBC and the points D, X and Y. By (i), (ii) and using the assumption $\frac{BY}{YC} = 3$,

$$\frac{BY}{YC} \cdot \frac{CX}{XM} \cdot \frac{MD}{DB} = 3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1.$$

Hence, by the converse of Menelaus' theorem, D, X and Y are collinear. □

Girard Desargues (1591–1661), a French architect, discovered an important and interesting result relating the collinearity of points and concurrency of lines on two triangles, which became a fundamental result in Projective Geometry. We shall now state this result and prove it by applying both Menelaus' theorem and its converse.

Desargues' Theorem.

Let ABC and $A'B'C'$ be two given triangles such that the lines AA', BB' and CC' are concurrent, as shown in Figure 8. Let X, Y and Z be, respectively, the points of intersection of the lines AB and $A'B'$, BC and $B'C'$ and CA and $C'A'$. Then X, Y and Z are collinear.

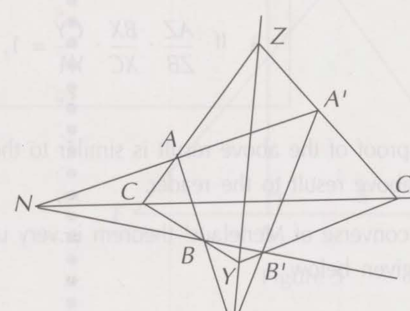

Figure 8

Proof

Observe that X, Y and Z are points on the lines formed from the sides AB, BC and CA of ΔABC respectively. Thus, to show that X, Y and Z are collinear, by the converse of Menelaus' theorem, it is enough to show that

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=1.$$

First, consider ΔNAB and its transversal $A'B'X$. By Menelaus' theorem,

$$\frac{NA'}{A'A}\cdot\frac{AX}{XB}\cdot\frac{BB'}{B'N}=1. \qquad \text{(i)}$$

Next, consider ΔNBC and its transversal $YB'C'$. By Menelaus' theorem,

$$\frac{NB'}{B'B}\cdot\frac{BY}{YC}\cdot\frac{CC'}{C'N}=1. \qquad \text{(ii)}$$

Now, consider ΔNCA and its transversal $Z'A'C'$. By Menelaus' theorem,

$$\frac{NC'}{C'C}\cdot\frac{CZ}{ZA}\cdot\frac{AA'}{A'N}=1. \qquad \text{(iii)}$$

Finally, the product of (i), (ii) and (iii) gives

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=1,$$

as was to be shown. □

We end this article by giving the following final example, which is actually Question 3 of the 1989 Asian Pacific Mathematics Olympiad. (Ten students from Singapore took part in this competition. Seven of them, Lam Vui Chiap, Lee Mun Yew, Loh Ngai Seng, Ng Lup Keen, Yan Weide, Yeo Don and Yeoh Yong Yeow, managed to solve this question completely. The common feature of their solutions was the use of Menelaus' theorem. We present here an outline of one of these approaches. The reader is invited to fill in any gaps.)

Example 4

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, and $A_5 = A_2$. For $n = 1$, 2 and 3, suppose that B_n is the midpoint of A_nA_{n+1} and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Solution

Our aim is to compute the values of $\frac{(D_1D_2D_3)}{(A_1A_2A_3)}$ and $\frac{(E_1E_2E_3)}{(A_1A_2A_3)}$, from which we can immediately determine the value of $\frac{(D_1D_2D_3)}{(E_1E_2E_3)}$.

Consider $\Delta A_2A_3B_1$ and its transversal $A_1D_1C_2$ (see Figure 9). By Menelaus' theorem,

$$\frac{A_2C_2}{C_2A_3}\cdot\frac{A_3D_1}{D_1B_1}\cdot\frac{B_1A_1}{A_1A_2}=1. \qquad \text{(i)}$$

As $\frac{A_2C_2}{C_2A_3}=\frac{1}{3}$ and $\frac{B_1A_1}{A_1A_2}=\frac{1}{2}$, it follows from (i) that

$$B_1D_1=\frac{1}{6}A_3D_1,$$

and so $B_1D_1=\frac{1}{7}A_3B_1$. (ii)

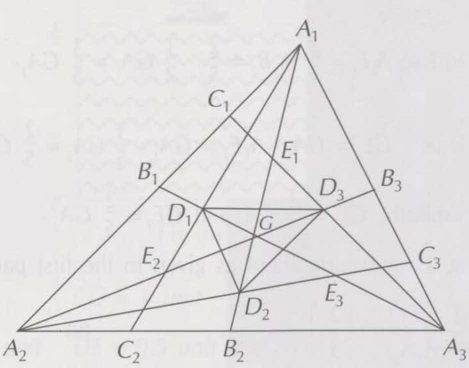

Figure 9

Let G denote the centroid of $\Delta A_1A_2A_3$; then

$$GB_1=\frac{1}{3}A_3B_1. \qquad \text{(iii)}$$

Thus $GD_1 = GB_1 - B_1D_1$

$$= \left(\frac{1}{3} - \frac{1}{7}\right) A_3B_1 \quad \textit{(by (ii) and (iii))}$$

$$= \frac{4}{21} A_3B_1$$

$$= \frac{4}{21} \cdot \frac{3}{2} GA_3 \quad \textit{(by (iii))}$$

$$= \frac{2}{7} GA_3,$$

i.e., $GD_1 = \frac{2}{7} GA_3$. (iv)

Likewise, $GD_2 = \frac{2}{7} GA_1$ (v)

and $GD_3 = \frac{2}{7} GA_2$. (vi)

It follows from (iv) and (v) that

$$\Delta GD_1D_2 \sim \Delta GA_3A_1,$$

and so

$$\frac{(GD_1D_2)}{(GA_3A_1)} = \left(\frac{GD_1}{GA_3}\right)^2 = \left(\frac{2}{7}\right)^2 = \frac{4}{49}. \quad \text{(vii)}$$

Likewise,

$$\frac{(GD_2D_3)}{(GA_1A_2)} = \frac{(GD_3D_1)}{(GA_2A_3)} = \frac{4}{49} \quad \text{(viii)}$$

Combining (vii) and (viii) yields

$$\frac{(D_1D_2D_3)}{(A_1A_2A_3)} = \frac{4}{49}. \quad \text{(ix)}$$

Next, consider $\Delta A_1A_2B_2$ and its transversal $A_3E_1C_1$. By Menelaus' theorem,

$$\frac{A_1C_1}{C_1A_2} \cdot \frac{A_2A_3}{A_3B_2} \cdot \frac{B_2E_1}{E_1A_1} = 1.$$

As $\frac{A_1C_1}{C_1A_2} = \frac{1}{3}$ and $\frac{A_2A_3}{A_3B_2} = 2,$

we have $A_1E_1 = \frac{2}{3} B_2E_1,$

and so $A_1E_1 = \frac{2}{5} A_1B_2 = \frac{2}{5} \cdot \frac{3}{2} GA_1 = \frac{3}{5} GA_1.$

Thus $GE_1 = GA_1 - A_1E_1 = GA_1 - \frac{3}{5} GA_1 = \frac{2}{5} GA_1.$

Similarly, $GE_2 = \frac{2}{5} GA_1$ and $GE_3 = \frac{2}{5} GA_3.$

Following a similar argument as given in the first part, we have

$$\frac{(E_1E_2E_3)}{(A_1A_2A_3)} = \left(\frac{2}{5}\right)^2 = \frac{4}{25}. \quad \text{(x)}$$

Combining (ix) and (x) yields

$$\frac{(D_1D_2D_3)}{(E_1E_2E_3)} = \frac{25}{49}.$$

M²

Mr Hang Kim Hoo obtained his BSc with Honours in Mathematics from NUS and MEd from NTU. His research interest lies in the teaching of Geometry. He has many years of experience in teaching mathematics at secondary schools and is currently a Specialist Inspector for Mathematics at the Ministry of Education. He has been a member of the International Mathematics Olympiad Training Committee since 1990.

Professor Koh Khee Meng obtained his first degree from Nanyang University in 1968 and PhD from Manitoba, Canada, in 1971. He then returned to teach at Nanyang University and he has been with the Department of Mathematics of NUS since 1980. He was the Chairman of the Singapore Mathematical Olympiad Training Committee from 1991 to 1993 and he was awarded the Faculty of Science Mathematics Teaching Award in 1994, 1995 and 1996.

■

Reference

[1] Hang Kim Hoo and Koh Khee Meng, *On Ceva's Theorem,* Mathematical Medley 23(1)(1996), 19-23.