by Wong Yan Loi .... **GEOMETRIC**

Introduction

In this note, we shall describe how various quantities of a triangle such as the sine or cosine of the three angles, the lengths of the three sides, or the lengths of the three altitudes etc., can be the roots of a cubic polynomial equation. By Newton's formulas, any symmetric polynomial of the three roots can be expressed in terms of the coefficients of the cubic polynomial. From this, one can obtain many interesting identities and inequalities relating various quantities of a triangle.

Let ABC be a triangle. Denote the length of BC, the length of CA and the length of AB by a, b and c respectively. Let r be the inradius, R the circumradius and s the semi-perimeter of ABC. Then the following results can be proved.

(1) a, b and c are the roots of the equation
$X^3 - 2sX^2 + (s^2 + r^2 + 4Rr)X - 4sRr = 0.$

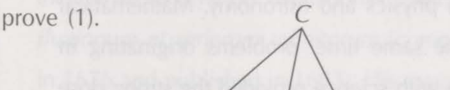
(2) $\sin A$, $\sin B$ and $\sin C$ are the roots of the equation
$4R^2X^3 - 4RsX^2 + (s^2 + r^2 + 4Rr)X - 2sr = 0.$

(3) $\cos A$, $\cos B$ and $\cos C$ are the roots of the equation
$4R^2X^3 - 4R(R+r)X^2 + (s^2 + r^2 - 4R^2)X + (2R+r)^2 - s^2 = 0.$

(4) $\tan A$, $\tan B$ and $\tan C$ are the roots of the equation
$(s^2 - (2R+r)^2)X^3 - 2srX^2 + (s^2 - r^2 - 4Rr)X - 2sr = 0.$

(5) The lengths of the three altitudes are the roots of the equation
$2RX^3 - (s^2 + r^2 + 4Rr)X^2 + 4s^2rX - 4s^2r^2 = 0.$

Let us prove (1).



Using the identities

$$a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2} \text{ and}$$

$$s - a = r\cot\frac{A}{2}, \text{ we have}$$

$$\sin^2\frac{A}{2} = \sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2} = \frac{a}{4R}\,\frac{r}{s-a} = \frac{ar}{4R(s-a)},$$

$$\cos^2\frac{A}{2} = \sin\frac{A}{2}\cos\frac{A}{2}\cot\frac{A}{2} = \frac{a}{4R}\,\frac{s-a}{r} = \frac{a(s-a)}{4Rr}.$$

From this, we obtain $\dfrac{ar}{4R(s-a)} + \dfrac{a(s-a)}{4Rr} = 1,$

which is equivalent to $a^3 - 2sa^2 + (s^2 + r^2 + 4Rr)a - 4sRr = 0.$

Similarly, b and c can be proved to satisfy the same condition. Therefore, a, b and c are the roots of the equation in (1).

It follows from (1) that two noncongruent triangles cannot have the same inradius, circumradius and semi-perimeter. Therefore, the three quantities r, R and s uniquely determine a triangle.

Using Newton's formulas which give the relations between roots and coefficients of polynomial equations, we obtain immediately from the equation in (1) that

(6) $a + b + c = 2s$, $ab + bc + ca = s^2 + r^2 + 4Rr$ and $abc = 4sRr$.

From these relations, any symmetric polynomial in a, b and c can be expressed in terms of r, R and s. For instance, one can easily derive the identities

$$a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr) \text{ and}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{s^2 + r^2 + 4Rr}{4sRr}$$

Equilateral Triangles

Consider the expression $(a-b)^2 + (b-c)^2 + (c-a)^2$. We have

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= 2(s^2 - 3r^2 - 12Rr).$$

Hence for any triangle ABC, $s^2 \geq 3r(r + 4R)$ and equality holds if and only if ABC is an equilateral triangle. Furthermore, by Euler's inequality (see [2, p.29]), we have $R \geq 2r$ and equality holds if and only if ABC is an equilateral triangle. Therefore, $s^2 \geq 3r(r+8r) = 27r^2$. Consequently, $s \geq 3\sqrt{3}\,r$ and equality holds if and only if ABC is an equilateral triangle. Since the area A of ABC is given by rs, this also gives an *isoperimetric** inequality $s^2 \geq 3\sqrt{3}A$.

IDENTITIES OF A TRIANGLE

*For a simple closed curve C of length L, bounding a region of area A in the plane, an inequality of the form $L^2 \geq kA$, where k is a positive constant, is called an isoperimetric inequality. It is well-known that $L^2 \geq 4\pi A$, with equality if and only if C is a circle.

Right-Angled Triangles

A characterization of right-angled triangles can be derived from (3). By (3), we have the relation $\cos A \cos B \cos C = \frac{1}{4R^2}[s^2 - (2R + r)^2]$. Therefore ABC is an acute-angled triangle, a right-angled triangle, or an obtuse-angled triangle according to $s > 2R + r$, $s = 2R + r$ or $s < 2R + r$. In particular, ABC is a right-angled triangle if and only if $s = 2R + r$.

Isosceles Triangles

Let's consider (1) again. We can obtain a condition for isosceles triangles. We have

$$[(a - b)(b - c)(c - a)]^2$$

$$= 18(a + b + c)(ab + bc + ca)(abc) - 27(abc)^2$$

$$+ (a + b + c)^2(ab + bc + ca)^2 - 4(a + b + c)^3(abc)$$

$$- 4(ab + bc + ca)^3$$

$$= 4r^2[-s^4 + 2(2R^2 + 10Rr - r^2)s^2 - r(4R + r)^3],$$

by applying (6). It follows from this that

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + r(4R + r)^3 = 0$$

if and only if ABC is an isosceles triangle.

Furthermore, since $[(a - b)(b - c)(c - a)]^2 \geq 0$, we have

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + r(4R + r)^3 \leq 0,$$

which is equivalent to

$$(2R^2 + 10Rr - r^2) - 2\sqrt{2(R - 2r)^3} \leq s^2$$

$$\leq (2R^2 + 10Rr - r^2) + 2\sqrt{R(R - 2r)^3}.$$

Hence, for any triangle ABC, s^2 is always between $2R^2 + 10Rr - r^2 \pm 2\sqrt{R(R - 2r)^3}$. This is called the fundamental inequality of the triangle. It can be proved (see [3, Chapter 1]) that for any positive numbers s, R and r, there exists a triangle with semi-perimeter s, circumradius R and inradius r if and only if $R \geq 2r$ and s^2 is between $2R^2 + 10Rr - r^2 \pm 2\sqrt{R(R - 2r)^3}$.

Note that for any triangle ABC,

$$2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3} \geq (2R + r)^2.$$

Therefore, when $s^2 = (2R^2 + 10Rr - r^2) + 2\sqrt{R(R - 2r)^3}$, ABC is an acute-angled isosceles triangle.

Gerretsen's Inequalities

The fundamental inequality though important, is not easy to apply due to the presence of the square root term $2\sqrt{R(R - 2r)^3}$. However, it can be weakened to two useful inequalities. Since

$$2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3}$$

$$= 4R^2 + 4Rr + 3r^2 - [R - 2r - \sqrt{R(R - 2r)}]^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and}$$

$$2R^2 + 10Rr - r^2 - 2\sqrt{R(R - 2r)^3}$$

$$= 16Rr - 5r^2 + [R - 2r - \sqrt{R(R - 2r)}]^2 \geq 16Rr - 5r^2,$$

we have $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$. These are the famous Gerrestsen's inequalities [3].

Next, we shall prove the useful inequality $\sqrt{3}s \leq r + 4R$ by means of another symmetric polynomial of a, b and c. We have

$$(a + b - c)^2(a - b)^2 + (b + c - a)^2(b - c)^2 + (c + a - b)^2(c - a)^2$$
$$= 2[(a + b + c)^4 - 5(a + b + c)^2(ab + bc + ca) + 6(a + b + c)abc + 4(ab + bc + ca)^2]$$
$$= 8r^2[(r + 4R)^2 - 3s^2].$$

Since

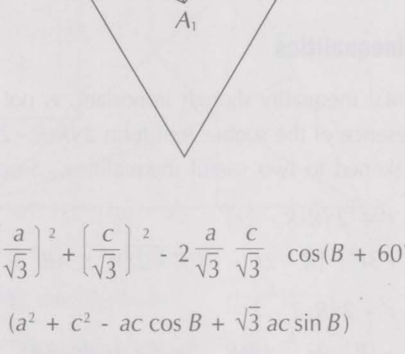
$$(a + b - c)^2(a - b)^2 + (b + c - a)^2(b - c)^2 + (c + a - b)^2(c - a)^2 \geq 0,$$

we have $\sqrt{3}s \leq r + 4R$ and equality holds if and only if ABC is an equilateral triangle.

Using Euler's inequality, we have $\sqrt{3}s \leq \frac{R}{2} + 4R = \frac{9R}{2}$. Consequently, $s \leq \frac{3\sqrt{3}}{2}R$ and equality holds if and only if ABC is an equilateral triangle.

Napoleon Triangles

Suppose that on each of the three sides of a triangle ABC, an equilateral triangle is erected outside ABC. Then the centroids of these equilateral triangles form a triangle, called the outer Napoleon triangle. If the equilateral triangles are erected inside ABC, then the resulting triangle formed by the three centroids is called the inner Napoleon triangle. It is well known (see [2, p.63]) that for any triangle ABC, both its outer and inner Napoleon triangles are equilateral triangles. In our context, the length of a side of the Napoleon triangle can be expressed as a symmetric polynomial of a, b and c, giving a simple proof of this result. Consider the case of the outer Napoleon triangle. Let A_1, B_1, and C_1 be the centroids of the equilateral triangles erected on the side BC, AC and AB of triangle ABC respectively. By cosine rule,



$A_1C_1^2 = \left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{c}{\sqrt{3}}\right)^2 - 2\frac{a}{\sqrt{3}}\frac{c}{\sqrt{3}}\cos(B + 60°)$. Hence

$$A_1C_1^2 = \frac{1}{3}(a^2 + c^2 - ac\cos B + \sqrt{3}\,ac\sin B)$$
$$= \frac{1}{3}\left[a^2 + c^2 - \frac{1}{2}(a^2 + c^2 - b^2) + \sqrt{3}\,ac\left(\frac{b}{\sqrt{2}R}\right)\right]$$
$$= \frac{1}{3}\left[\frac{1}{2}(a^2 + b^2 + c^2) + \frac{\sqrt{3}}{2R}abc\right]$$
$$= \frac{1}{3}(s^2 - r^2 - 4Rr + 2\sqrt{3}sr)$$
$$= \frac{1}{3}[(s^2 + \sqrt{3}r)^2 - 4r(r + R)],$$

which is a constant. Similarly, $B_1C_1^2$ and $A_1B_1^2$ equal to this constant. Consequently, $A_1B_1C_1$ is an equilateral triangle of side

$$\frac{1}{\sqrt{3}}[(s + \sqrt{3}r)^2 - 4r(r + R)]^{\frac{1}{2}}.$$

For the case of the inner Napoleon triangle, one can similarly show that it is an equilateral triangle of side

$$\frac{1}{\sqrt{3}}[(s - \sqrt{3}r)^2 - 4r(r + R)]^{\frac{1}{2}}.$$

The Brocard Angle

The Brocard angle of a triangle (see [1]) is the angle ω defined by $\cot\omega = \cot A + \cot B + \cot C$. Since it is a symmetric function of the three angles, it can be expressed easily in terms of s, R and r. From (4),

$$\cot\omega = \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C} = \frac{s^2 - r^2 - 4Rr}{2sr}.$$

Since $s^2 \geq 3r(r + 4R)$, we have

$$\cot\omega \geq \frac{3r(r + 4R) - r^2 - 4Rr}{2sr} = \frac{r + 4R}{s}.$$

Using the inequality $\sqrt{3}s \leq r + 4R$, we deduce that $\cot\omega \geq \sqrt{3}$. Consequently, the Brocard angle of any triangle is less than or equal to 30°.

Geometric Inequalities

Any symmetric polynomial expression of the lengths of the three sides, sine, cosine or tangent of the three angles etc., can be expressed in terms of R, r and s. Using Gerretsen's inequalities and others, many interesting geometric inequalities can be obtained. In this section, we give some examples in this respect.

(a) $\cos A\cos B\cos C \leq \frac{1}{8}$.

To prove this, we first have

$$\cos A\cos B\cos C = \frac{1}{4R^2}[s^2 - (2R + r)^2] \text{ by (3).}$$

Using Gerretsen's inequality, $s^2 \leq 4R^2 + 4Rr + 3r^2$ and then Euler's inequality, we have

$$\cos A\cos B\cos C$$
$$= \frac{1}{4R^2}[s^2 - (2R + r)^2] \leq \frac{1}{4R^2}[4R^2 + 4Rr + 3r^2 - (2R + r)^2]$$
$$= \frac{1}{2}\left(\frac{r}{R}\right)^2 \leq \frac{1}{8}.$$

Moveover, equality holds if and only if ABC is an equilateral triangle.

(b) $\dfrac{b+c}{a} + \dfrac{c+a}{b} + \dfrac{a+b}{c} \geq 6.$

This can be proved by using the other Gerretsen's inequality, $s^2 \geq 16Rr - 5r^2$ and also Euler's inequality.

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$$

$$= \frac{(a+b+c)(ab+bc+ca) - 3abc}{abc}$$

$$= \frac{s^2 + r^2 - 2Rr}{2Rr} \geq \frac{(16Rr - 5r^2) + r^2 - 2Rr}{2Rr}$$

$$= \frac{7R - 2r}{R} \geq 6.$$

(c) $\dfrac{c}{(s-a)(s-b)} + \dfrac{a}{(s-b)(s-c)} + \dfrac{b}{(s-c)(s-a)} \geq \dfrac{2\sqrt{3}}{r}.$

This can be proved as follows. By the inequality $\sqrt{3}s \leq r + 4R$, we have

$$\frac{c}{(s-a)(s-b)} + \frac{a}{(s-b)(s-c)} + \frac{b}{(s-c)(s-a)} = \frac{2(4R+r)}{sr} \geq \frac{2\sqrt{3}}{r}.$$

Interested readers can derive their own geometric inequalites based on the same principle. M^2

References

[1] Chan, S. C., *The Review of the Penguin Dictionary of Curios and Interesting Geometry* by David Wells, Mathematical Medley, **23**, No. 1, March (1996), p33-34.

[2] Coxeter, H. S. M., Greitzer, S. L., *Geometry Revisited*, New Mathematical Library, **19**, MAA (1967).

[3] Mitrinović D. S., Pečarić J. E., Volenec, V., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers (1989).

Dr Wong Yan Loi graduated from the University of Hong Kong with a BSc degree and an MPhil degree in mathematics. He received his PhD in 1988 from the University of California at Berkeley. He is now a senior lecturer of mathematics at NUS.