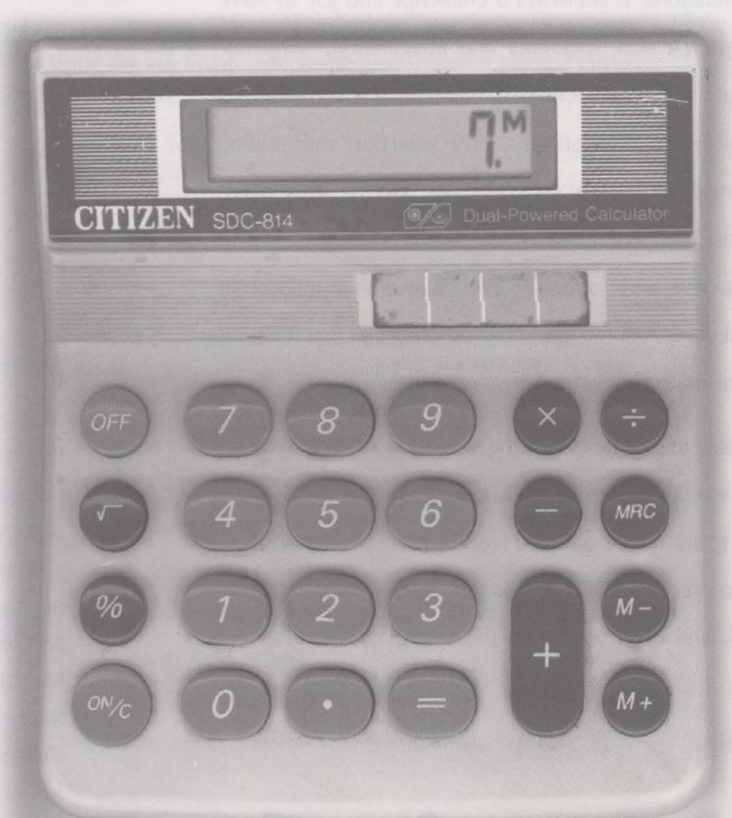
COUNTING

- Its Principles and Techniques (4) -

by K. M. Koh and B. P. Tan



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9. Distribution of Identical Objects into Distinct Boxes

In Section 7 of [3], we have discussed the problem of distributing identical objects into distinct boxes. Let us consider this problem again.

Figure 9.1 shows 7 distinct boxes into which 5 identical objects are to be distributed.

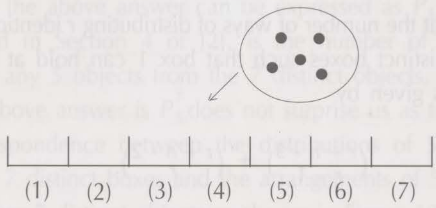


Figure 9.1

Question 1

Suppose that each box can hold at most one object. In how many ways can this be done ?

As shown in Figure 9.2, we first select 5 boxes from the seven, and then put an object in each of the 5 selected boxes.

There are $\binom{7}{5}$ ways for the first step, and 1 way for the second. Thus the answer is $\binom{7}{5}$.

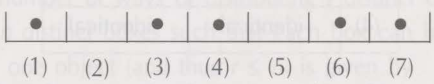


Fig 9.2

Question 2

Suppose that each box can hold any number of objects. In how many ways can this be done?

Questions of this sort were studied in Section 7. Applying the method of counting certain binary sequences and bijection principle (BP), we obtain the answer to question 2 which is given by $\binom{5+7-1}{5}$, i.e. $\binom{11}{5}$.

Question 3

Suppose now that there are 11 identical objects (instead of 5) to be put into 7 distinct boxes. In how many ways can this be done if each box must hold at least one object?

To fulfil the requirement that no box is empty, we first put one object in each box. We are now left with 4 objects, but are free to put them in any box. Again, by what we learned in Section 7, the answer is given by $\binom{4+7-1}{4}$, i.e., $\binom{10}{4}$.

Let us summarize what we discussed above by stating the following general results

Suppose that there are *r* identical objects to be distributed into *n* distinct boxes.

(I) If each box can hold at most one object, (thus $r \le n$), the number of ways of distribution is given by $\binom{n}{r}$.

(II) If each box can hold any number of objects, the number of ways of distribution is given by $\binom{r+n-1}{r}$.

(III) If each box must hold at least one object (thus $r \ge n$), the number of ways of distribution is given by $\binom{(r-n)+n-1}{r-n}$, i.e., $\binom{r-1}{r-n}$.

Example 9.1

Eight letters are to be selected from the five English vowels *a, e, i, o,* u with repetition allowed. In how many ways can this be done if

(i) there are no other restrictions?

(ii) each vowel must be selected at least once?

(i) Some examples of ways of selection are give below:

(1) *a, a, u, u, u, u, u, u;*

(2) *e, i, i, o, o, o, u, u;*

(3) *a, e, i, i, o, o, u, u.*

As shown in Figure 9.3, these selections can be treated as ways of distributing 8 identical objects into 5 distinct boxes.

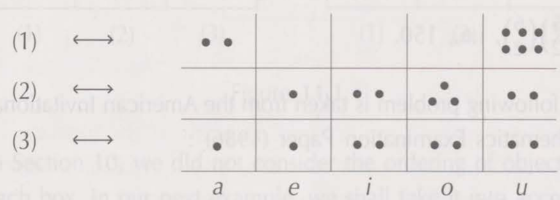


Figure 9.3

Thus, by (*BP*) and result (II) above, the number of ways of selection is given by $\binom{8+5-1}{8}$, i.e, $\binom{12}{4}$.

(ii) As shown in the last box of Figure 9.3, a way of selection which includes each vowel can be treated as a way of distribution such that no box is empty. Thus, by (*BP*) and result (III) above, the number of ways of selection is given by $\binom{(8-5)+5-1}{8-5}$, i.e, $\binom{7}{3}$.

Example 9.2

Consider the following two 13-digit binary sequences:

1 1 1 0 1 0 1 1 0 0 0 0,

1 0 0 0 1 1 0 0 1 1 1 1 0.

For binary sequences, any block of 2 adjacent digits is of the form : 00, 01, 10, 11. In each of the above sequences, there are three 00, two 01, three 10 and four 11. Find the number of 13-digit binary sequences which have exactly three 00, two 01, three 10 and four 11.

To have exactly three 10 and two 01 in a sequence, such a sequence must begin with '1', end with '0', and have the changeovers of '1' and '0' as shown below, where each of the boxes (1), (3) and (5) [resp., (2), (4) and (6)] holds only '1' (resp., '0') and at least one '1' (resp., '0').

| 1 | 0 | 1 | 0 | 1 | 0 |

(1) [10] (2) [01] (3) [10] (4) [01] (5) [10] (6)

For instance, the two sequences given in the problem are of the form :

| 111 | 0 | 1 | 0 | 111 | 0000 |

| 1 | 000 | 11 | 00 | 1111 | 0 |

To have three 00 and four 11 in such a sequence, we must (i) put three more '0' in boxes (2), (4) or (6) arbitrarily and (ii) put four more '1' in boxes (1), (3) or (5) arbitrarily. (Check that there are 13 digits altogether.) The number of ways to do (i) is $\binom{3+3-1}{3}$, i.e., $\binom{5}{2}$ while that of (ii) is $\binom{4+3-1}{4}$, i.e., $\binom{6}{2}$. Thus, by (MP), the number of such sequences is $\binom{5}{2}\binom{6}{2}$, i.e., 150.

The following problem is taken from the American Invitational Mathematics Examination Paper (1986) :

Problem 9.1

In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc.. We denote these by *TH, HH*, etc.. For example, in the sequence *HHTTHHHHTHHTTTT* of 15 coin tosses we observe that there are five *HH*, three *HT*, two *TH* and four *TT* subsequences. How many different sequences of 15 coin tosses will contain exactly two *HH*, three *HT*, four *TH* and five *TT* subsequences?

Problem 9.2

Consider the following two 15-digit ternary sequences (formed by 0, 1 and 2) :

0 0 0 1 1 1 2 2 0 0 1 1 2 2 2

0 1 2 2 2 0 0 0 0 1 1 1 1 2 2

Observe that each of the sequences contains exactly three 00, three 11, three 22, two 01, two 12 and one 20. Find the number of such ternary sequences.

Problem 9.3

Find the number of ways of distributing 10 identical tables into 5 distinct rooms in each of the following cases.

(i) Room 1 holds at most 2 tables.

(ii) Each of room 1 and room 2 holds at most 1 table.

Problem 9.4

Show that the number of ways of distributing *r* identical objects into *n* distinct boxes such that box 1 can hold at most one object is given by

$$\binom{r+n-3}{r-1} + \binom{r+n-2}{r}.$$

10. Distribution of Distinct Objects into Distinct Boxes

As can be seen from the various examples given in Sections 7 and 9, the *distribution problem,* which deals with the counting of ways of distributing objects into boxes, is a basic model for many counting problems. In distribution problem, objects can be identical or distinct, and boxes too can be identical or distinct. Thus there are four cases to be considered; namely

	Objects	Boxes
(1)	identical	distinct
(2)	distinct	distinct
(3)	distinct	identical
(4)	identical	identical

We have considered case (1) in Sections 7 and 9. Cases (3) and (4) will be discussed in due course. In this section, we shall consider Case(2). Before we proceed, we would like to point out that the ordering of the distinct objects in each box is not taken into consideration in the discussion in this section. Suppose that 5 distinct balls are to be put into 7 distinct boxes.

Question 1 In how many ways can this be done if each box can hold at most one ball?

Question 2 In how many ways can this be done if each box can hold any number of balls?

We first consider Question 1. As shown in Figure 10.1, let *a, b, c, d* and *e* denote the 5 distinct balls. First, we put '*a*' (say)

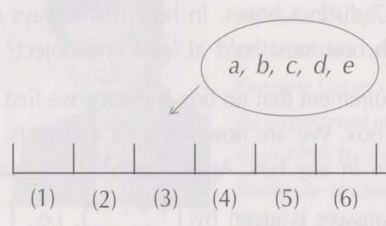


Figure 10.1

into one of the boxes. There are 7 choices. Next, we consider 'b' (say). As each box can hold at most one ball, and one of the boxes is occupied by 'a', there are now 6 choices for 'b'. Likewise, there are, respectively, 5, 4 and 3 choices for 'c', 'd' and 'e'. Thus, by (*MP*), the number of ways of distribution is given by $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

Note that the above answer can be expressed as P^7_5 , which as defined in Section 4 of [2], is the number of ways of arranging any 5 objects from the 7 distinct objects. The fact that the above answer is P^7_5 does not surprise us as there is a 1-1 correspondence between the distributions of 5 distinct balls into 7 distinct boxes and the arrangements of 5 distinct objects from 7 distinct objects as shown in Figure 10.2. (Find out the rule of the correspondence!)

$\{a, b, c, d, e\}$ $\qquad\qquad$ $\{1, 2, 3, 4, 5, 6, 7\}$

(1)	(2)	(3)	(4)	(5)	(6)	(7)		
b	c		a	e		d	$\longleftrightarrow$	41275
e	d	c	b			a	$\longleftrightarrow$	74321

Figure 10.2

In general, we have :

> The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold at most one object (and thus $r \le n$) is given by
> $$P^n_r \ (= \frac{n!}{(n-r)!}).$$

We now consider Question 2. There are 7 ways of putting 'a' in the boxes. As each box can hold any number of balls, there are also 7 choices for each of the remaining balls b, c, d, and e. Thus, by (*MP*), the answer is 7^5.

In general, we have :

> The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold any number of objects is given by n^r .

Problem 10.1

Find the number of ways for a teacher to distribute 6 different books to 9 students if

(i) there is no restriction;

(ii) no student gets more than one book.

Problem 10.2

Let A be the set of ways of distributing 3 distinct objects into 4 distinct boxes 1, 2, 3 and 4 with no restriction, and let B be the set of 3-digit numbers using 1, 2, 3, 4 as digits with repetition allowed (e.g., 222, 441, 431,...). Establish a 1-1 correspondence between A and B.

Problem 10.3

Suppose that m distinct objects are to be distributed into n distinct boxes so that each box contains at least one object (thus $m \ge n$). In how many ways can this be done if

(i) $m = n$? $\quad$ (ii) $m = n + 1$? $\quad$ (iii) $m = n + 2$?

Problem 10.4

Find the number of ways of distributing 8 distinct objects into 3 distinct boxes if each box must hold at least 2 objects.

11. Other Variations

Two cases of distribution problem were discussed in the preceding sections. In this section, we shall study some of their variations.

When *identical* objects are put in distinct boxes, whether the objects in each box are *ordered* or not makes no difference. The situation is no longer the same if the objects are distinct as shown in Figure 11.1

$d\ e$	$a\ b\ c$		$\neq$	$e\ d$	$c\ b\ a$	
(1)	(2)	(3)		(1)	(2)	(3)

Figure 11.1

In Section 10, we did not consider the ordering of objects in each box. In our next example, we shall take it into account.

Example 11.1

Suppose that 5 distinct objects a, b, c, d, e are distributed into 3 distinct boxes, and that the ordering of objects in each box counts. In how many ways can this be done ?

First, consider 'a' (say), clearly, there are 3 choices of a box for 'a' to be put in (say, 'a' is put in box 2). Next, consider 'b'. 'b' can be put in one of the 3 boxes. The situation is different if 'b' is put in box 2 due to the existence of 'a' in that box. As the ordering of objects in each box counts, if 'b' is put in box 2, then there are two choices for 'b', namely,

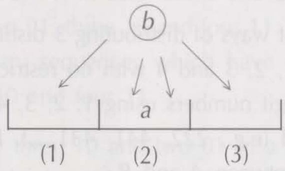


Figure 11.2

left of 'a' or right of 'a' as indicated in Figure 11.2. Thus, altogether, there are 4 choices for 'b', (say, 'b' is put in box 3).

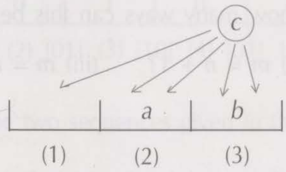
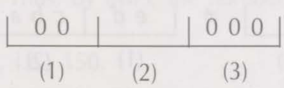


Figure 11.3

Now, consider 'c'. As shown in Figure 11.3, 'c' has 5 choices. Continuing in this manner, we see that 'd' and 'e' have, respectively, 6 and 7 choices. Thus the answer is given by $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

Let us try a different approach to solve the above problem. First, we pretend that the objects *a, b, c, d* and *e* are all identical. The number of ways of distributing 5 identical objects into 3 distinct boxes is, by result (II) in Section 9, $\binom{5+3-1}{5}$, i.e., $\binom{7}{2}$. Next, take such a way of distribution,

say 

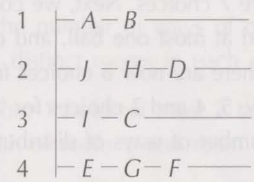
Since the 5 objects are actually distinct and the ordering of objects in each box counts, such a distribution for *identical* objects corresponds to 5! different distributions for *distinct* objects. Thus, by (*MP*), the answer is given by $\binom{7}{2} \cdot 5!$, which agrees with the first answer $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

In general, we have:

> The number of ways of distributing *r distinct* objects into *n distinct* boxes such that the ordering of objects in each box counts is given by
>
> $$\binom{r+n-1}{r} \cdot r!$$
>
> which is equal to
>
> $$n(n+1)(n+2) \ldots (n+r-1).$$

Problem 11.1

Ten students are to line up in a 4-line queue as shown below. In how many ways can this be done?



Problem 11.2

Solve Problem 10.3 under an additional condition that the ordering of objects in each box counts.

In our previous discussion on distribution problem, objects are either *all identical* or *all distinct*. We now consider a case that is a mixture of these two.

Example 11.2

Four identical objects 'a', three identical objects 'b' and two identical objects 'c' are to be distributed into 9 distinct boxes so that each box contains one object. In how many ways can this be done?

First, consider 4 *a*'s (say). Among the 9 boxes, we choose 4 of them, and put one 'a' in each box chosen. Next, consider 3 *b*'s (say). Among the 5 boxes that remain, we choose 3, and put one 'b' in each box chosen (see Figure11.4). Finally, we put a 'c' in each of the 2 remaining boxes.

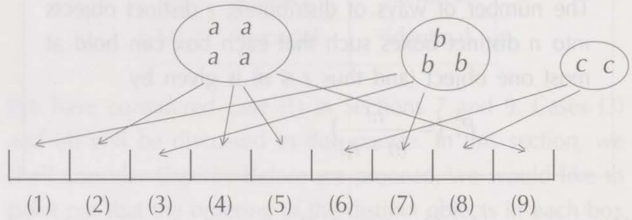


Figure 11.4

There are $\binom{9}{4}$ ways for step 1, $\binom{5}{3}$ ways for step 2 and $\binom{2}{2}$ (= 1) way for step 3. Thus, by (*MP*), the answer is given by

$$\binom{9}{4}\binom{5}{3} \cdot 1 = \frac{9!}{4!5!} \cdot \frac{5!}{3!2!} = \frac{9!}{4!3!2!}$$

Remark

In the above, 'a' is considered first, followed by 'b' and 'c'. The answer is independent of the order. For instance, if 'b' is considered first, followed by 'c' and 'a', by applying a similar argument, we arrive at $\binom{9}{3}\binom{6}{2}\binom{4}{4}$, which is again $\frac{9!}{4!3!2!}$.

There is a 1-1 correspondence between the distributions considered in Example 11.2 and the arrangements of 4*a*'s, 3*b*'s and 2*c*'s in a row as shown in Figure 11.5.

| a | c | a | b | a | b | c | b | a | $\longleftrightarrow$ *a c a b a b c b a*

| c | b | a | a | b | a | a | b | c | $\longleftrightarrow$ *c b a a b a a b c*

Figure 11.5

Thus, by the result of Example 11.2, the number of arrangements of 4 *a*'s, 3*b*'s and 2*c*'s in a row is given by

$$\frac{9!}{4!\,3!\,2!}$$

In general, we have:

> Suppose that there are n_1 identical objects of type 1, n_2 identical objects of type 2, … , n_k identical objects of type *k*. Let $n = n_1 + n_2 + \dots + n_k$. Then the number of arrangements of these *n* objects in a row is given by
>
> $$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k},$$
>
> which is equal to
>
> $$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Problem 11.3

Show that

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\cdots n_k!},$$

where $n = n_1 + n_2 + \dots + n_k$.

Let us re-consider Example 11.1. We observe that there is a 1-1 correspondence between the distributions considered in Example 11.1 and the arrangements of *a, b, c, d, e* and 2 1's as shown in Figure 11.6. By the above result, the number of

(1)	(2)	(3)	
be	a	dc	$\longleftrightarrow$ be | a | dc
cde		ba	$\longleftrightarrow$ cde | | ba

Figure 11.6

arrangements of *a, b, c, d, e* and 2 1's is given by $\frac{7!}{2!}$, which agrees also with the first two answers.

Problem 11.4

Find the number of arrangements of 3 *x*'s, 4 *y*'s and 5 *z*'s in a row if

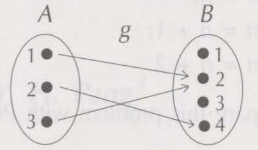
(i) there is no restriction;

(ii) no two *y*'s are adjacent;

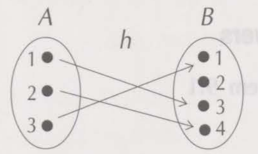
(iii) any two *x*'s are separated by at least 2 other letters.

Let $A = \{1, 2, \dots, m\}$ and $B = \{1, 2, \dots, n\}$, where $m, n \ge 1$. A *mapping* f from *A* to *B*, denoted by $f: A \to B$, is a rule which assigns to each element *a* of *A* a *unique* element $f(a)$ of *B*. For instance, given $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, the following rules *g*, *h* defined, respectively, by

$$\begin{cases} g(1) = 2 \\ g(2) = 4 \\ g(3) = 2 \end{cases}$$

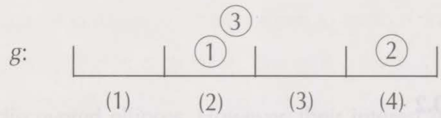


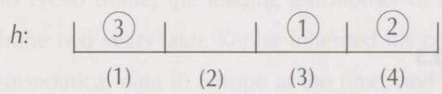
$$\begin{cases} h(1) = 3 \\ h(2) = 4 \\ h(3) = 1 \end{cases}$$



are mappings from *A* to *B*. A mapping $f: A \to B$ is *injective* (i.e., 1-1) if $f(i) \ne f(j)$ in *B* whenever $i \ne j$ in *A*. Thus the mapping *h* defined above is 1-1 while *g* is not (why?).

A mapping $f: A \to B$ can actually be regarded as a way of distributing *m* distinct objects 1, 2, … *m* into *n* distinct boxes 1, 2, … , *n* (ordering of objects in each box does not count) in the following manner : $f(i) = j$ means that object '*i*' is put in box '*j*'. Thus, the mappings *g* and *h* defined above can be treated as the ways of distribution shown below:





Problem 11.5

Let $A = \{1, 2, \dots, m\}$ and $B = \{1, 2, \dots, n\}$, where $m, n \ge 1$. Find

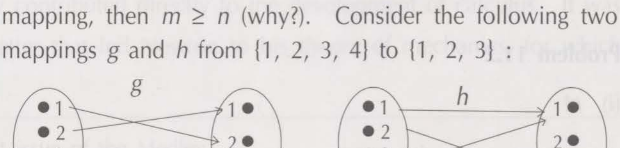
(i) the number of mappings from *A* to *B*;

(ii) the number of 1-1 mappings from *A* to *B* (here, $m \le n$);

(iii) the number of mappings $f: A \to B$ such that $f(i) < f(j)$ in *B* whenever $i < j$ in *A* (here, $m \le n$);

(iv) the number of mappings $f: A \to B$ such that $f(1) = 1$.

A mapping $f: A \to B$ is *surjective* (i.e., onto) if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$. The mappings *g* and *h* from {1, 2, 3} to {1, 2, 3, 4} defined above are not surjective. Indeed, if $f: \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is an onto mapping, then $m \ge n$ (why?). Consider the following two mappings *g* and *h* from {1, 2, 3, 4} to {1, 2, 3} :



It can be checked that *g* is onto while *h* is not so.

Problem 11.6

Let $A = \{1, 2, \dots, m\}$ and $B = \{1, 2, \dots, n\}$. Find the number of onto mappings from A to B in each of the following cases:

(i) $m = n$;

(ii) $m = n + 1$;

(iii) $m = n + 2$.

(Compare this problem with Problem 10.3.) M²

Answers

Problem 9.1

560

Problem 9.2

64

Problem 9.3

(i) $\binom{13}{3} + \binom{12}{3} + \binom{11}{3}$

(ii) $\binom{12}{2} + 2\binom{11}{2} + \binom{10}{2}$

Problem 10.1

(i) 9^6

(ii) $\frac{9!}{3!}$

Problem 10.2

The i^{th} digit is 'j' when and only when object 'i' is put in box 'j'.

Problem 10.3

(i) $n!$

(ii) $\binom{n+1}{2} \cdot n!$

(iii) $\binom{n+2}{2}\binom{n}{2} \cdot n! + \binom{n+2}{3} \cdot n!$

Problem 10.4

$\left[\binom{8}{4}\binom{4}{2} + \binom{8}{3}\binom{5}{3}\right] \cdot 3!$

Problem 11.1

$\frac{13!}{3!}$

Problem 11.2

(i) $n!$

(ii) $\binom{n+1}{2} \cdot n! \cdot 2!$

(iii) $\binom{n+2}{2}\binom{n}{2} \cdot n! \cdot 2! \cdot 2! + \binom{n+2}{3} \cdot n! \cdot 3!$

Problem 11.4

(i) $\frac{12!}{3!4!5!}$

(ii) $\binom{9}{4}\frac{8!}{3!5!} = \binom{9}{4}\binom{8}{3}$

(iii) $\binom{8}{3}\frac{9!}{4!5!} = \binom{9}{4}\binom{8}{3}$

Problem 11.5

(i) n^m

(ii) $\frac{n!}{(n-m)!}$

(iii) $\binom{n}{m}$

(iii) n^{m-1}

Problem 11.6

same as Problem 10.3

References

[1] K. M. Koh and B. P. Tan, *Counting -Its Principles and Techniques (1)*, Mathematical Medley Vol 22 March (1995) 8-13.

[2] K. M. Koh and B. P Tan, *Counting - Its Principles and Techniques (2)*, Mathematical Medley Vol 22 September (1995) 47-51.

[3] K. M. Koh and B. P Tan, *Counting - Its Principles and Techniques (3)*, Mathematical Medley Vol 23 March (1996) 9-14