On CEVA'S THEOREM

by Hang Kim Hoo & Koh Khee Meng

Many of the secondary pupils are aware of the following two facts in geometry:

(1) If D, E and F are the midpoints of the sides BC, CA and AB respectively of ΔABC (see Figure 1), then the line segments AD, BE and CF (called the medians of ΔABC) meet at a common point. We say that the medians of ΔABC are *concurrent*.

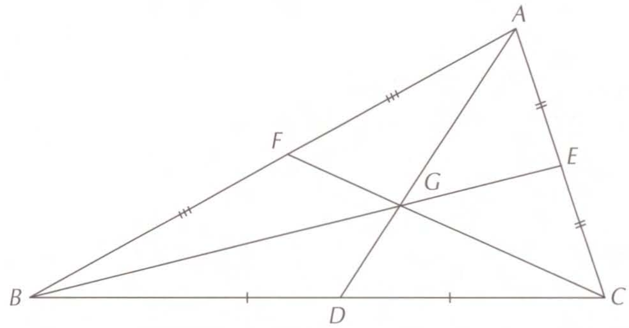

Figure 1

(2) Suppose that, on the other hand, D and E are the midpoints of BC and CA respectively. Join A and D, and B and E, and assume that AD and BE meet at S as shown in Figure 2. Join C and S, and extend CS to meet AB at F. Then is F the midpoint of AB.

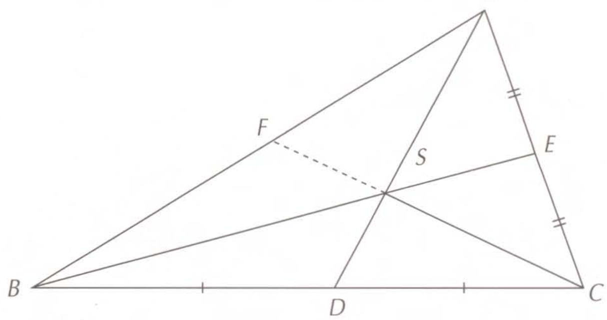

Figure 2

In this article, we shall introduce a famous and important result in geometry which generalizes the facts mentioned above. This result is known as Ceva's theorem, in honour of the Italian mathematician Giovanni Ceva (1648-1734) who published it in 1678.

THE THEOREM

In a triangle ABC, any line segment joining a vertex to a point on its opposite side (extended if necessary) is called a *cevian* of ΔABC. Figure 3 shows three cevians AP, BQ and CR. Suppose that they are concurrent. What can be said about the relationship among the six line segments AR, RB, BP, PC, CQ and QA?

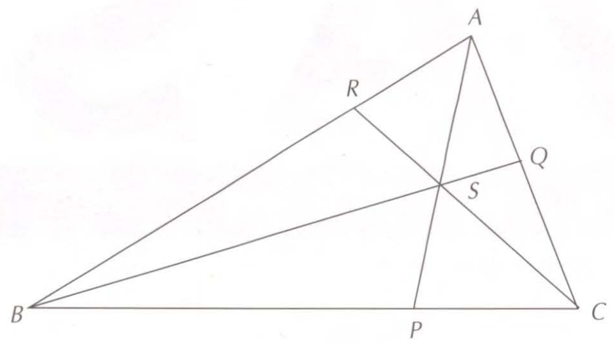

Figure 3

Ceva answered this question by establishing the following beautiful result.

Ceva's Theorem If the cevians AP, BQ and CR of ΔABC are concurrent, then

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1 \qquad (1)$$

There are different proofs of this theorem. The proof which we are going to present here makes use of the notion of area. For this purpose, given ΔXYZ, we shall denote by (XYZ) its area.

Suppose that the cevians AP, BQ and CR meet at S as shown in Figure 3. We then observe that

$$\frac{AR}{RB}=\frac{(ACR)}{(BCR)}=\frac{(ASR)}{(BSR)}.$$

Thus $\quad AR\cdot(BCR) = RB\cdot(ACR) \qquad (2)$

and $\quad AR\cdot(BSR) = RB\cdot(ASR) \qquad (3)$

(2) and (3) give

$$AR\cdot((BCR) - (BSR)) = RB\cdot((ACR) - (ASR)),$$

which implies that

$$\frac{AR}{RB}=\frac{(ACS)}{(BCS)} \qquad (4)$$

Likewise, we have

$$\frac{BP}{PC}=\frac{(BAS)}{(CAS)} \qquad (5)$$

and

$$\frac{CQ}{QA}=\frac{(CBS)}{(ABS)} \qquad (6)$$

From (4), (5) and (6) we obtain

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1$$

as required.

Let us show an application of Ceva's theorem.

Example 1 In Figure 4, the cevians AD, BE and CF of ΔABC meet at P. Given that $2BD = 3DC$, $3AE = 4EC$ and $(APF) = 72$, find the area (BPF).

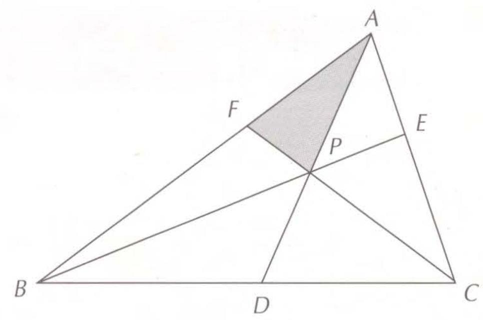

Figure 4

As AD, BE and CF are concurrent, by Ceva's theorem, we have

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1.$$

Thus, by assumption

$$\frac{AF}{FB}\cdot\frac{3}{2}\cdot\frac{3}{4}=1,$$

and so $\quad \dfrac{AF}{FB}=\dfrac{8}{9}.$

Since $\quad \dfrac{AF}{FB}=\dfrac{(APF)}{(BPF)},$

$$(BPF) = (APF)\cdot\frac{FB}{AF} = 72\cdot\frac{9}{8} = 81.$$

THE CONVERSE

Ceva's theorem states that if the three cevians of ΔABC shown in Figure 3 are concurrent, then equality (1) holds. Does the converse of Ceva's theorem hold? That is, if P, Q and R are points on the sides BC, CA and AB respectively such that equality (1) holds, are then the cevians AP, BQ and CR always concurrent? The answer is in the affirmative as shown below.

The Converse of Ceva's Theorem If P, Q and R are points on the sides BC, CA and AB of ΔABC respectively such that

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1 \qquad (7)$$

then AP, BQ and CR are concurrent.

To prove this result, suppose that the cevians AP and BQ meet at S. Join C and S, and extend CS to meet AB at R' as shown in Figure 5. Our aim is to show that $R' = R$.

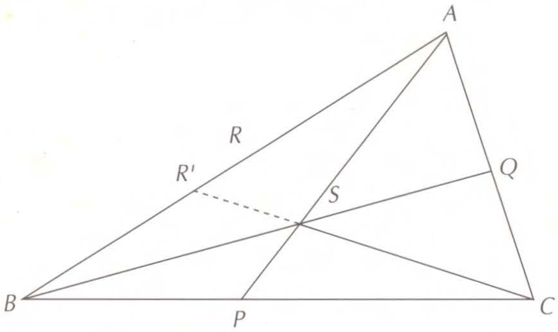

Figure 5

Since AP, BQ and CR' are concurrent, by Ceva's theorem, we have

$$\frac{AR'}{R'B} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1 \qquad (8)$$

It follows from (7) and (8) that

$$\frac{AR'}{R'B} = \frac{AR}{RB}$$

which in turn implies that R and R' coincide. This proves that AP, BQ and CR are concurrent.

We note that Ceva's theorem and its converse are also valid even if a cevian joins a vertex to a point on its opposite side extended as shown in Figure 6.

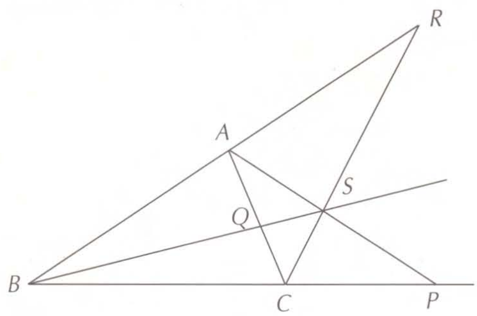

Figure 6

At the beginning of this article, we pointed out that the three medians of a triangle are always concurrent. We shall now see that this result is an immediate consequence of the converse of Ceva's theorem. Indeed, as shown in ΔABC of Figure 1, we have $AF = FB$, $BD = DC$ and $CE = EA$, and so

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Thus, the medians AD, BE and CF are concurrent by the converse of Ceva's theorem. We call this common point (point G in Figure 1) the *centroid* of ΔABC. The centroid is one of the most important points associated with a triangle. In what follows, we shall introduce another two important points associated with a triangle.

Example 2 In ΔABC of Figure 7, the cevians AP, BQ and CR are perpendicular to BC, CA and AB respectively. They are called the *altitudes* of ΔABC. We shall show by the converse of Ceva's theorem that the three altitudes are concurrent.

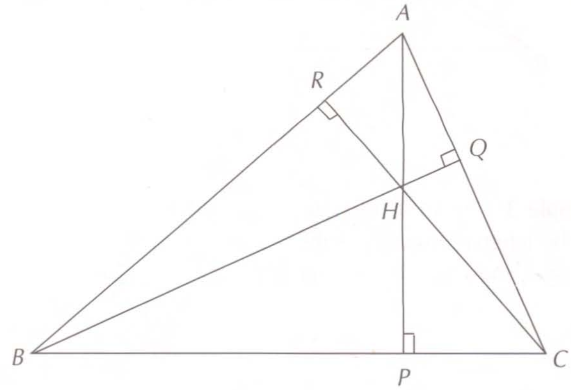

Figure 7

Consider the right-angled ΔARC. We have:

$$\cos A = \frac{AR}{CA},$$

i.e., $\qquad AR = CA\cos A.$

Likewise, we have

$$\begin{aligned} RB &= BC \cos B, \\ BP &= AB \cos B, \\ PC &= CA \cos C, \\ CQ &= BC \cos C, \\ \text{and} \quad QA &= AB \cos A. \end{aligned}$$

Thus

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{CA \cos A}{BC \cos B} \cdot \frac{AB \cos B}{CA \cos C} \cdot \frac{BC \cos C}{AB \cos A} = 1.$$

Hence, by the converse of Ceva's theorem, the altitudes AP, BQ and CR are concurrent. We call this common point (point H of Figure 7) the *orthocentre* of ΔABC.

Before we proceed to introduce another 'centre' of a triangle, let us recall a formula for the area of a triangle. Consider ΔABC of Figure 8. To find the area (ABC), we draw the altitude BY as shown. Now

$$(ABC) = \frac{1}{2} CA.BY$$

and $\qquad BY = AB \sin A.$

Thus we have

$$(ABC) = \frac{1}{2} CA.AB \sin A \qquad (9)$$

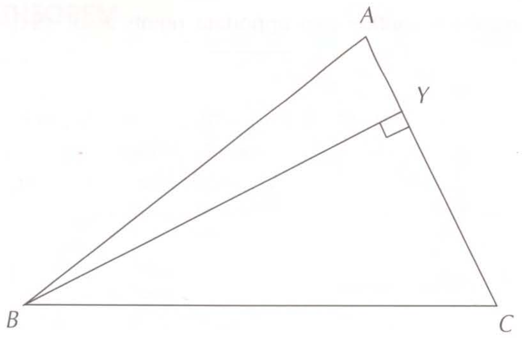

Figure 8

Example 3 In ΔABC of Figure 9, the cevians AX, BY and CZ are the internal bisectors of the angles A, B and C respectively. We shall show that these three cevians are concurrent.

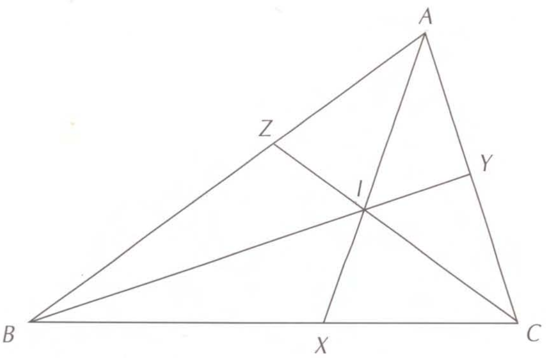

Figure 9

Observe that

$$\frac{AZ}{ZB} = \frac{(ACZ)}{(BCZ)}$$

and by (9),

$$(AZC) = \frac{1}{2} \, . \, CA \, . \, CZ \, . \, \sin\left(\frac{C}{2}\right),$$

$$(BZC) = \frac{1}{2} \, . \, BC \, . \, CZ \, . \, \sin\left(\frac{C}{2}\right).$$

Thus $$\frac{AZ}{ZB} = \frac{(AZC)}{(BZC)} = \frac{CA}{BC}$$

i.e., $$\frac{AZ}{ZB} = \frac{CA}{BC}.$$

Similarly, $$\frac{BX}{XC} = \frac{AB}{CA}$$

and $$\frac{CY}{YA} = \frac{BC}{AB}.$$

It follows that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{CA}{BC} \cdot \frac{AB}{CA} \cdot \frac{BC}{AB} = 1.$$

Hence, by the converse of Ceva's theorem, the internal angle bisectors AX, BY and CZ are concurrent. This common point (point I of Figure 9) is called the *incentre* of ΔABC.

We shall show another application of the converse of Ceva's theorem.

Example 4 In the parallelogram $ABCD$ of Figure 10, E, F, G and H are points on AB, BC, CD and DA respectively such that $EG \parallel BC$ and $HF \parallel AB$. Let P be the point of intersection of EG and HF. Show that the lines AF, CE and DP are concurrent.

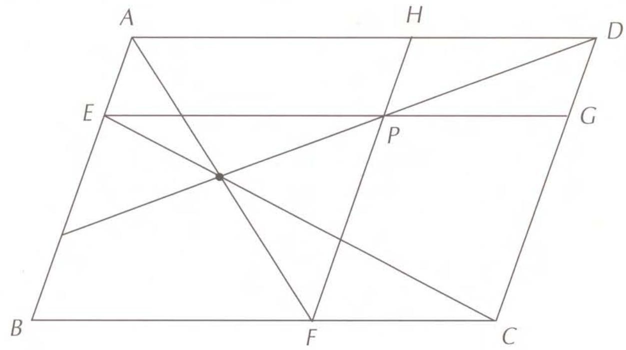

Figure 10

Join E and F, and extend DP to meet EF and AB at L and K respectively as shown in Figure 11. Let N be the point of intersection of EG and AF, and M the point of intersection of CE and HF.

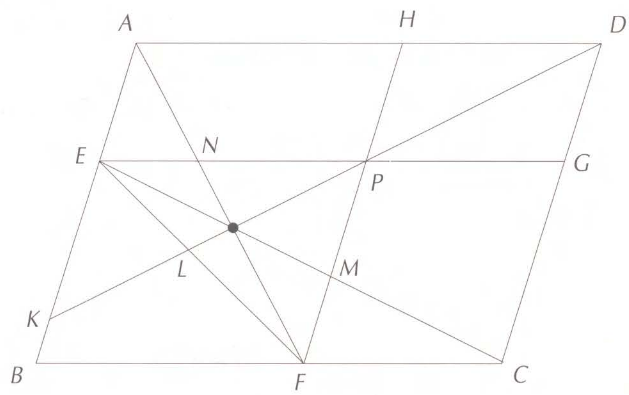

Figure 11

Observe that EM, FN and PL are three cevians of ΔEFP, and to show that AF, CE and DP are concurrent is the same as to show that the cevians EM, FN and PL of ΔEFP are concurrent.

Note that

$$\frac{FM}{MP} = \frac{CF}{EP} \qquad (\Delta FMC \sim \Delta PME)$$

$$= \frac{GP}{EP}$$

$$= \frac{GD}{EK} \qquad (\Delta DPG \sim \Delta KPE)$$

$$= \frac{AE}{EK}$$

$$\frac{PN}{NE} = \frac{PF}{EA} \qquad (\Delta PNF \sim \Delta ENA)$$

$$= \frac{EB}{EA},$$

and

$$\frac{EL}{LF} = \frac{EK}{PF} \qquad (\Delta ELK \sim \Delta FLP)$$

$$= \frac{EK}{EB}.$$

Thus

$$\frac{FM}{MP} \cdot \frac{PN}{NE} \cdot \frac{EL}{LF} = \frac{AE}{EK} \cdot \frac{EB}{EA} \cdot \frac{EK}{EB} = 1.$$

By the converse of Ceva's theorem, *EM, FN* and *PL* (and hence *CE, AF* and *DP*) are concurrent.

We shall now consider our final example.

Example 5 In a circle *C* with centre *O* and radius *r*, let C_1, C_2 be two circles with centres O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangential to *C* at A_i so that C_1, C_2 are externally tangential to each other at *A* (see Figure 12). Prove that the three lines OA, O_1A_2 and O_2A_1 are concurrent.

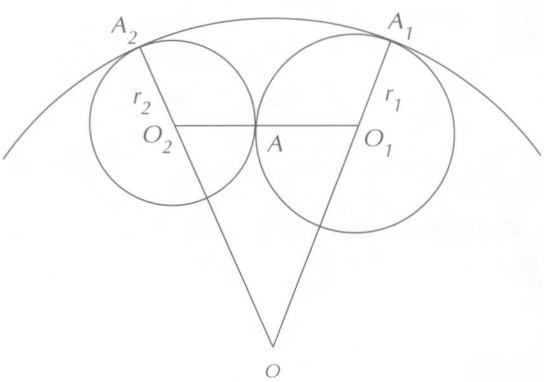

Figure 12

The problem given in Example 5 is actually Question 2 of the 1992 Asian Pacific Mathematical Olympiad. Pang Siu Taur, a secondary student then, took part in the competition and gave a short proof of this problem by applying the converse of Ceva's theorem. We shall now present his proof.

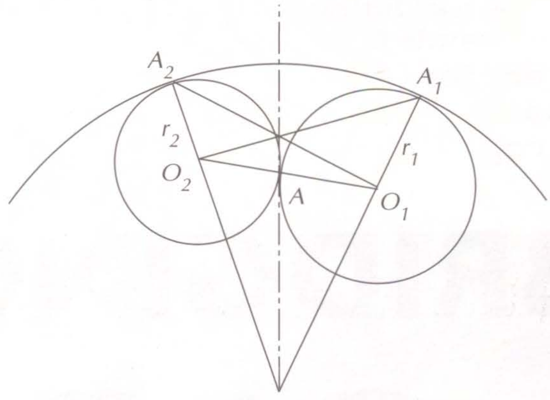

Figure 13

As shown in Figure 13, consider the cevians OA, O_1A_2 and O_2A_1 of ΔOO_1O_2.

Observe that

$$\frac{OA_1}{A_1O_1} \cdot \frac{O_1A}{AO_2} \cdot \frac{O_2A_2}{A_2O} = \frac{r}{r_1} \cdot \frac{r_1}{r_2} \cdot \frac{r_2}{r} = 1.$$

Thus, by the converse of Ceva's theorem, OA_1, O_1A_2 and O_2A_1 are concurrent. M^2

Mr Hang Kim Hoo obtained his BSc with Honours in Mathematics from NUS and MEd from NTU. His research interest lies in the teaching of Geometry. He has many years of experience in teaching mathematics at secondary schools and is currently a Specialist Inspector for Mathematics at the Ministry of Education. He has been a member of the International Mathematics Olympiad Training Committee since 1990.

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