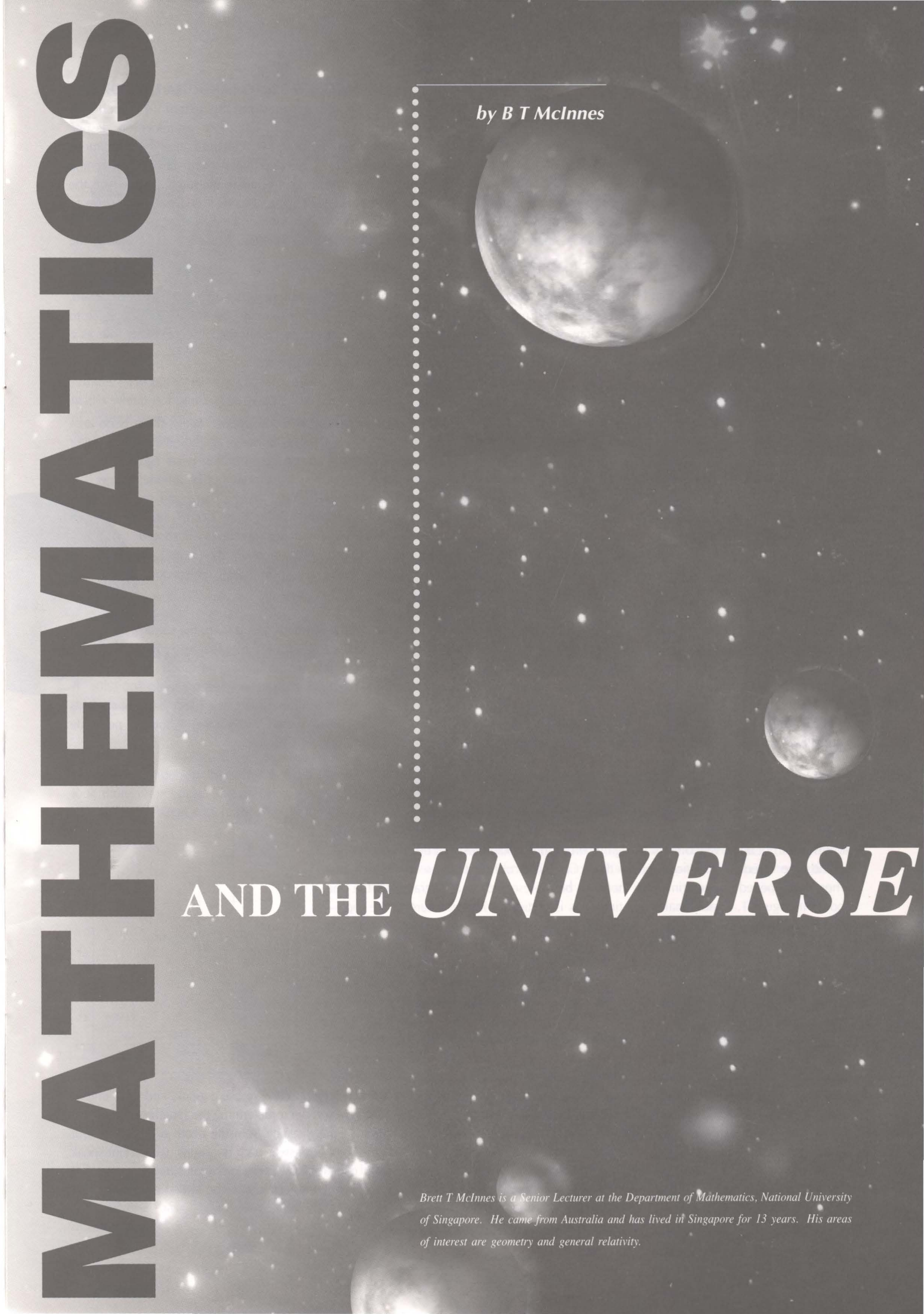
MATHEMATICS AND THE *UNIVERSE*

by B T McInnes



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We live in an era of confusion. It may be true that today the level of public awareness of the basic ideas of science is higher than ever before; but one is frequently tempted to question the extent to which this public understands the marvels presented to it. Not long ago I watched an American "talk show" on television, despite my natural aversion to these embarrassing spectacles. I made an exception in this case because the "celebrity guest" was a famous scientist, an authority on ants and (consequently) on human behaviour. One certainly feels gratified that a scientist can now become a celebrity — dare we hope that this will one day be the case in Singapore? Alas, however, the inane questions inflicted on the great man by the audience would suffice to make even the most hardened schoolteacher cringe. Again, it is always agreeable to note that almost every large bookstore contains numerous books on black holes and related topics. One's pleasure is sometimes abated, however, upon finding these works classified under "Occult"; the fact that the Bible is often similarly located affords me only a little consolation.

I am glad to be able to report that I have never found a book on mathematics sharing a shelf with manuals on geomancy or Satanism, though once, in Bangkok, a salesgirl suggested that I might seek such works in the "Self Improvement" section. This brings me to the main point of this article : I want to draw your attention to the ability of mathematics to clarify the obscure, to its role as a solvent of mystification.

THE UNNECESSARILY MYSTERIOUS COSMOS

To many people, Cosmology — the study of the entire Universe — is the most interesting of the sciences. Sadly, there is no area of science (with the obvious exception of evolutionary biology) which has generated more nonsensical commentary or sheer confusion. Everyone is vaguely familiar with the basic "facts": the Universe "began" ten or fifteen billion years ago in a great "explosion" called the Big Bang; the Universe is still "expanding" away from that "point"; it may or may not be infinitely large; space and time themselves "came into being" at the Big Bang, so that time did not exist "before" the Big Bang; and so on. Some people are, quite rightly, dissatisfied with these "facts", however. Where, they ask, did the Big Bang occur? (It has been suggested — perhaps in jest, though, in this subject, one never knows — that a suitably imposing monument should be raised there to mark the birthplace of the Universe.) If the whole Universe "exploded from a point", then how can it be infinite if its age is finite? In short, most of the "well-known facts" about Cosmology are not well understood, and indeed not all of them are facts.

Regrettably, many people are quite content with this state of affairs: the mysteries of the Big Bang afford them the same kind of childish gratification as that enjoyed by believers in UFOs or by the students of Singapore's astonishingly high population of ghosts. Against this popular tide of obscurantism and folly I ,propose that we set something much less popular, namely mathematics. Now in a publication of this kind, the reader's sympathies are not in question; even so, however, I have to confess that the mathematics of Cosmology is not at all simple. We teach the subject to a hardy band of students at NUS: there are few complaints, even from these rugged individualists, that the course is excessively easy. I hope to persuade you, however, that even a little mathematics can clarify some basic points of confusion in Cosmology.

A STITCH IN SPACE-TIME

The basic framework for the subject is provided by the concept of *space-time*. This is simply the set of all *events* or happenings: the death of Hitler, your birth — of course, events need not be as welcome or as significant as these examples. An event is described by the time when it happened, and by three Cartesian coordinates x, y, z which tell us where it happened. Space-time can therefore be represented by a diagram like the one shown in Figure 1. Traditionally, the time axis is vertical, with the future at the top — we all like to feel that things are looking up. The entire history of a small object such as a human being is represented by a line which is more or less vertical, since not even Boris Becker can avoid aging altogether. (I have, of course, suppressed the y and z axes, hence the odd geography. Things may not be entirely accurate in the time direction either.) The line would be exactly straight and vertical only for a stationary object.

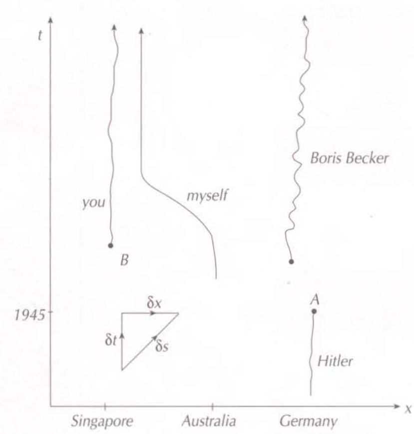


Figure 1

Now, of course Newton and indeed Descartes might have drawn such pictures to beguile their leisure hours, if any. These space-time diagrams are of little interest in themselves, because the underlying space seems to have no *geometry*. By this I mean the following. Given a vertical displacement in the diagram (see Figure 1), we know how to assign a length to it: we use a clock, and the "length" is δt. The definition of length in the horizontal direction is still more obvious. But how are we to determine the length of a *diagonal* displacement such as δs in the diagram? What is the distance AB in the diagram — the distance from the death of Hitler to your birth? (The reader will permit me to assume that he or she is not aged 51.) Unless we can answer such questions, we cannot speak of a geometry for space-time. In ordinary geometry, of course, we answer such questions by using Pythagoras' theorem, so that $\delta s = \sqrt{\delta t^2 + \delta x^2}$. But who can say whether the geometry of space-time is "ordinary"? This is a question to be settled by observation, not prescription.

MINKOWSKI AND THAT OTHER FELLOW

H. Minkowski

In 1908 the great mathematician (note this word) Minkowski realised that the facts about space and time discovered by Einstein a few years earlier could all be explained as follows. Minkowski declared that space-time does indeed have a geometry: the rule for measuring diagonal displacements is

$$\delta s = .\sqrt{|-\delta t^2 + \delta x^2|}$$

Notice the crucial minus sign, which necessitates the absolute value signs under the square root. Here we use light-years as our unit of length, and years for time, so that the speed of light is $|\frac{\delta s}{\delta t}| = 1$, and it follows that $\delta s = 0$ for any two events joined by a pulse of light. That may seem odd, but it simply reminds us that the geometry of space-time is not "ordinary".

A. Einstein

In 1916 Einstein went even farther and proposed that the geometry of space-time is not fixed — instead, it depends on the amount of matter present. He constructed a gadget (it isn't a number; you can think of it as being something like a matrix which is a function of space and time) which describes the geometry of space-time. The amount of matter present is measured by a similar gadget called the stress-energy-momentum tensor, T. Einstein's equation is

$$G = 8\pi T.$$

This is the fundamental equation of General Relativity; it is far more significant than the rather trivial relation $E = mc^2$.

At this point less indulgent readers are wondering what has become of the clarity supposedly brought to Cosmology by mathematics. Let me therefore proceed to a concrete solution of Einstein's equation, one which describes a whole Universe. Pray take note of the care with which I have chosen my words: I spoke of "*a*" solution. It is by no means the only one. In fact I have selected this one simply because it is easy to write it down.

WHERE PYTHAGORAS SLIPPED UP

A "solution" of Einstein's equation is of course a rule for measuring "diagonal" distances in space-time diagrams. The one we shall study looks like this:

$$(\delta s)^2 = |-(\delta t)^2 + (6\pi\rho_0)^{2/3} t^{4/3} [(\delta x)^2 + (\delta y)^2 + (\delta z)^2]|.$$

Here ρ_0 is the density of matter in the Universe at the present time, and I have restored δy and δz in order to make the formula look more impressive. The matter in this space-time is very simple: it consists of an infinite collection of galaxies (each regarded as a point in space) which are all at rest. The galaxies are therefore represented by vertical straight lines in the diagram (Figure 2).

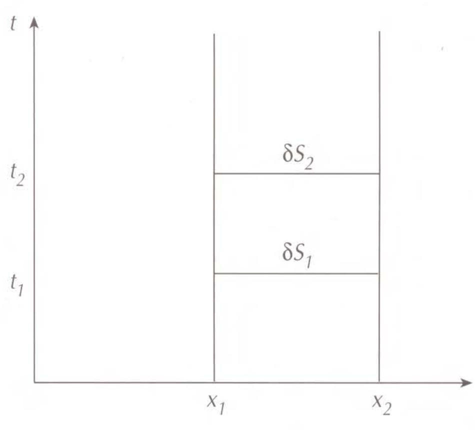


Figure 2

Now consider two galaxies as shown. The distance between them (indicated by the horizontal line) at time t_1 is found by setting $\delta t = 0$ in the above formula: we obtain

$$\delta s_1 = \delta s(t = t_1) = (6\pi\rho_0)^{1/3} t_1^{2/3} (x_2 - x_1).$$

Similarly,

$$\delta s_2 = \delta s(t = t_2) = (6\pi\rho_0)^{1/3} t_2^{2/3} (x_2 - x_1).$$

and so

$$\delta s_2 / \delta s_1 = (t_2 / t_1)^{2/3} > 1.$$

Thus, the distance has increased, the Universe has expanded. But wait! Did I not say that the galaxies are at rest? Indeed I did, and I say it again: the galaxies have not moved a millimetre — look at the diagram! What has changed is not the positions of the galaxies, but rather the geometry of the space between them. We now see that the word "expansion", which brings to mind an "outward motion", is not the right one; indeed, the notion that the "expansion" has a "direction" is quite wrong. Rather than speak of an "expanding" Universe, we should simply say that the Universe has a "dynamic geometry". More loosely, we can simply say that Pythagoras left something out, namely the function of time $t^{4/3}$.

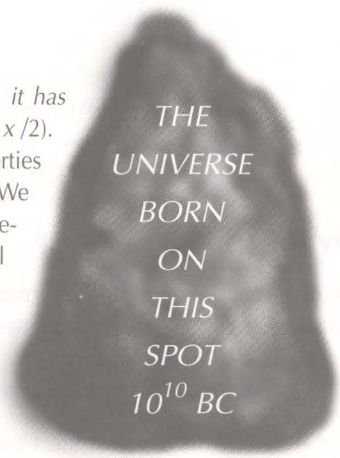
EXPANDING — BUT NOT MOVING

With the aid of really rather simple mathematics, we can now begin to sweep away some cosmic confusion. First, imagine a ray of light moving towards us from some very distant galaxy. As time goes by, the geometry changes, and the wavelength must increase, and so the light will look redder. That is, Einstein's theory predicts that distant galaxies will look redder than they actually are. This is indeed precisely what we see, the so-called "red shift", and it is direct evidence that the geometry of space is dynamic. (Nearly all books on Cosmology, including those written by people who ought to know better, state that the red shift is due to the "fact" that the galaxies are "rushing away from us". This is quite nonsensical of course — look at that diagram again!) Now the energy of light is inversely proportional to its wavelength, so the energy of light is always decreasing: you can easily convince yourself, using the above formulae, that if E_1 is the energy at time t_1, and E_2 is the energy at time t_2, then

$$E_1/E_2 = (t_2/t_1)^{2/3}$$

Now contemplate the truly mind-boggling consequences of this simple formula. As we go back in time, t_1 decreases towards zero, and E_1 becomes arbitrarily large. Long ago, then, the Universe must have been very hot, with extremely energetic light moving about in all directions. But what happens when $t_1 = 0$? Answer: nothing happens, because the above formula does not make sense at $t_1 = 0$. Again, all too many books on Cosmology declare that " E_1 is infinite" at $t_1 = 0$, but I am confident that the readers of the Medley would never be guilty of such an elementary error. *We* would never say that, for example, the function $|\log(x)|$ "equals infinity" at $x = 0$; instead, we say that this function is defined only on the domain $x > 0$. In the cosmological case, we simply have to admit that space-time geometry is defined only on $t > 0$. This conclusion is confirmed by geometrical calculations involving the so-called curvature tensor.

A BIG BANG IN THE PRIVACY OF YOUR HOME



Now allow me to remind you of a curious property of the set of real numbers $t > 0$: *it has no first element.* Given any positive number x, I can always find a smaller one (such as $x/2$). The Universe we have been studying therefore has the apparently contradictory properties that it is of finite age — and yet it has no beginning! What, then, of the Big Bang? We clearly *cannot* say that "$t = 0$ is the Big Bang", for, since "$t = 0$" is not a part of space-time, it is not an event, not a happening; and so "$t = 0$ is the Big Bang" would entail the rather bizarre conclusion that the Big Bang never happened. Instead, we have to say that the Big Bang is that part of space-time corresponding to $0 < t < \varepsilon$, where ε is some very small but non-zero real number. Our picture is now as shown in Figure 3, and a glance at it should dissolve many mysteries. (Here *A, B, C* are typical galaxies, and t_1 is any time.) For example, it is clear that the Big Bang did not occur at some special point; it occurred everywhere, and if you wish to erect a monument to it, your own living-room is quite as good a spot as any. Again, all talk of "explosions" is clearly out of place here. This particular solution of Einstein's equation corresponds to a Universe which is infinite in spatial extent (that is, x, and also y and z, can be arbitrarily large), but this should occasion no surprise, since the spatial extent has always (that is, for all $t > 0$) been infinite — this infinite space has *not* "exploded from a point". I leave it to the reader to solve his or her favourite cosmological mystery by referring to Figure 3.

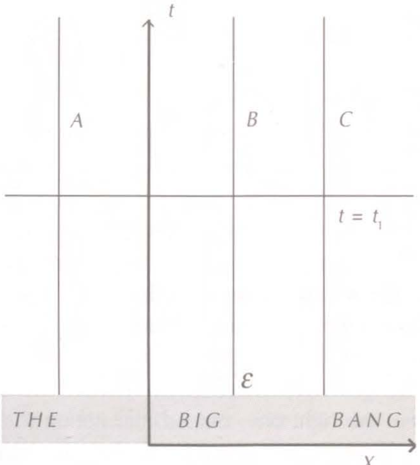


Figure 3

A CHALLENGE

Allow me to conclude, not with further celebrations of the clarifying power of mathematics, but with a challenge to test your understanding. You can show — it isn't obvious — that there are galaxies so far away from us that the light from them has not had time to reach us in the ten billion years or so since the Big Bang. Draw a space-time diagram to illustrate this situation. How far away from us are the most distant galaxies we can possibly see? (Hint: Use the fact that $\delta s = 0$ for a ray of light. The answer is *not* ten billion light years, by the way!)

FURTHER READING:

Flat and Curved Space-Times, GFR Ellis and RM Williams, Oxford University Press 1988.