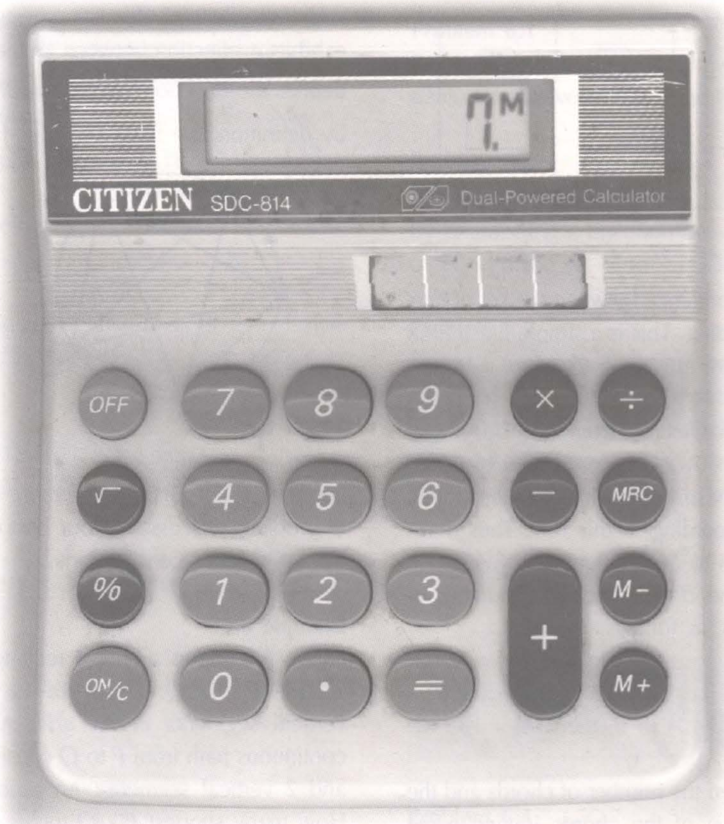
COUNTING

- Its Principles and Techniques (3) -

by K M Koh and B P Tan



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6. Bijection Principle

In Sections 2 and 3 of [1], we introduced two basic principles of counting, namely, the Addition Principle (AP) and the Multiplication Principle (MP). In this section we shall introduce another basic principle of counting, known as the Bijection Principle. With the help of the counting techniques introduced in [1] and [2], we shall learn some applications of the Bijection Principle in the present article.

Suppose that there are 120 parking spaces in a building, and a number of cars are parked within the building. Assume that each car occupies a space, and each space is occupied by a car (see Figure 6.1). Then we know that the number of cars in the building is 120 without having to count the cars one by one. The number of cars and the number of spaces are the same

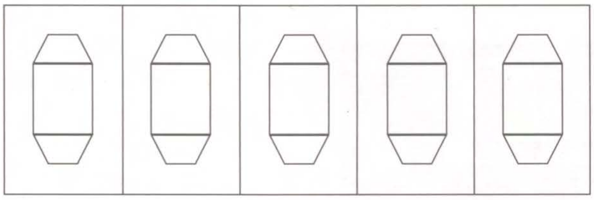


Figure 6.1

because there is a 1 - 1 correspondence between the set of cars and the set of spaces in the building. This is a simple illustration of the Bijection Principle that we are going to state.

Let A and B be two finite sets. A *bijection* from A to B is a rule which assigns to each element of A a unique element of B and at the same time for each element of B there is a unique element of A which is assigned to it according to this rule. Thus if we can construct a bijection from A to B, the elements of A and B will be paired off, so that A and B have the same number of elements. This is stated formally as follows:

> **The Bijection Principle (BP).** Let A and B be two finite sets. If there is a bijection from A to B, then $|A| = |B|$.

In Example 5.5 in [2], we counted the number of chords and the number of points of intersection of the chords joining some fixed points on the circumference of a circle. Let us study the problem again. Figure 6.2 shows five distinct points on the circumference of a circle.

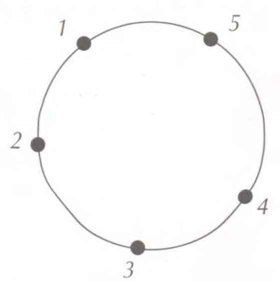


Figure 6.2

How many chords are there formed by these points? Let A be the set of such chords, and B the set of 2-element subsets of $\{1, 2, 3, 4, 5\}$. It is easy to see that the rule which assigns to each chord x in A the 2-element subset $\{\alpha, \beta\}$, where α, β are the two points which determine the chord x, is a bijection between A and B. Figure 6.3 shows the bijection. Thus by (BP), $|A| = |B|$. We have $|B| = \binom{5}{2}$. Hence $|A| = \binom{5}{2}$.

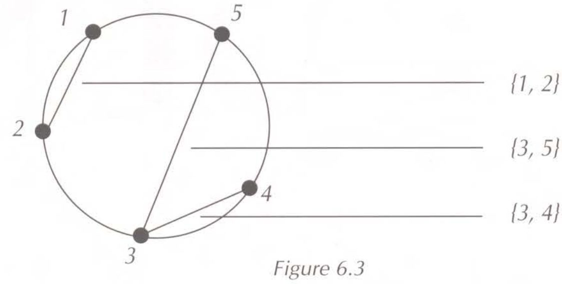


Figure 6.3

Next, how many points of intersection of these $\binom{5}{2}$ chords are there within the circle if no three of the chords are concurrent within the circle? Let A be the set of such points of intersection, and B the set of 4-element subsets of $\{1, 2, 3, 4, 5\}$. Figure 6.4 exhibits a bijection between A and B (figure out the rule which defines the bijection!). Thus by (BP), $|A| = |B|$. Since $|B| = \binom{5}{4}$ by definition, we have $|A| = \binom{5}{4}$.

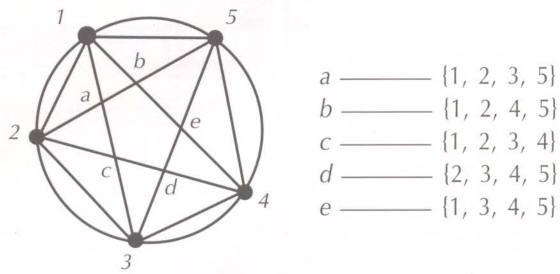


Figure 6.4

Let us proceed to see some more applications of (BP). The type of problems discussed in our next example can always be found in mathematics competitions.

Example 6.1
Figure 6.5 shows a 2 x 4 rectangular grid with two specified corners P and Q. There are 12 horizontal segments and 10 vertical segments in the grid. A *shortest* $P-Q$ route is a continuous path from P to Q consisting of 4 horizontal segments and 2 vertical segments. An example is shown in Figure 6.5. How many shortest P - Q routes are there in the grid?

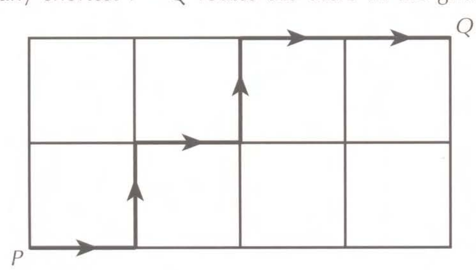
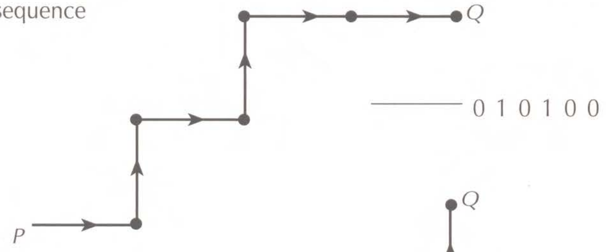


Figure 6.5

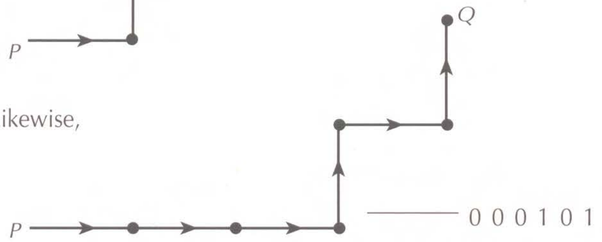
Certainly we can solve the problem directly by listing all the possible shortest routes. This, however, would not be practical if we wish to solve the same problem in, say, a 190 x 100 rectangular grid. We look for a more efficient way.

There are two types of segments: horizontal and vertical. Let us use a '0' to represent a horizontal segment, and a '1' to represent a vertical segment. Thus the shortest P - Q route shown in

Figure 6.5 can accordingly be represented by the following binary sequence



Likewise,



and so on. Let A be the set of all shortest P - Q routes, and B the set of 6-digit binary sequences with two 1's. Then one sees that the above way of representing a shortest P - Q route by a binary sequence in B establishes a bijection between the sets A and B. Thus by (BP), $|A| = |B|$. The counting of $|B|$ is easy. Indeed, $|B| = \binom{6}{2}$ (see Example 5.4 in [2]). Thus $|A| = \binom{6}{2}$.

Let us continue to discuss another typical counting problem.

Example 6.2
For a set S, let $P(S)$ denote the set of all subsets of S, inclusive of S and the empty set ø. Thus, for $\mathbb{N}_n = \{1, 2, \ldots, n\}$, $1 \le n \le 3$, we have

$$P(\mathbb{N}_1) = \{ø, \{1\}\},$$
$$P(\mathbb{N}_2) = \{ø, \{1\}, \{2\}, \{1, 2\}\},$$
$$P(\mathbb{N}_3) = \{ø, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Note that $|P(\mathbb{N}_1)| = 2$, $|P(\mathbb{N}_2)| = 4$, $|P(\mathbb{N}_3)| = 8$. Table 1 in Section 4 of [2] shows that $|P(\mathbb{N}_4)| = 16$. What is the value of $|P(\mathbb{N}_5)|$?

For convenience, let $A = P(\mathbb{N}_5)$. Thus A is the set of all subsets of {1, 2, 3, 4, 5}. Represent these subsets by 5-digit binary sequences as follows:

ø	———	00000
{1}	———	10000
{2}	———	01000
⋮		
{5}	———	00001
{1,2}	———	11000
⋮		
{4,5}	———	00011
⋮		
{1,3,5}	———	10101
⋮		
{1,2,3,4,5}	———	11111

The rule is that the ith digit of the corresponding binary sequence is '1' if 'i' is in the subset; and '0' otherwise. Let B be the set of all 5-digit binary sequences. Clearly, the above rule establishes a bijection between A and B. Thus by (BP), $|A| = |B|$. Since $|B| = 2^5$ (see Example 3.1 in [1]), $|A| = 2^5$.

Note that $|P(\mathbb{N}_1)| = 2 = 2^1$, $|P(\mathbb{N}_2)| = 4 = 2^2$, $|P(\mathbb{N}_3)| = 8 = 2^3$, $|P(\mathbb{N}_4)| = 16 = 2^4$, and now $|P(\mathbb{N}_5)| = 2^5$. What is $|P(\mathbb{N}_n)|$ for $n \ge 1$? See Problem 6.1.

The above examples show that (BP) is indeed a very powerful method of enumeration. In the course of applying (BP), we replace the less familiar set A by a more familiar set B, and transform the more difficult problem of counting $|A|$ to the easier problem (hopefully) of counting $|B|$. Although the members of A and B could be very much different in nature, as long as there is a bijection between A and B, we have $|A| = |B|$.

Problem 6.1
For each positive integer n, show that $|P(\mathbb{N}_n)| = 2^n$ by establishing a bijection between $P(\mathbb{N}_n)$ and the set of n-digit binary sequences.

Problem 6.2
Let n and k be integers with $1 \le k \le n$. Prove that $\binom{n}{k} = \binom{n}{n-k}$ by establishing a bijection between the set of k-element subsets of $\mathbb{N}_n$ and the set of $(n - k)$-element subsets of $\mathbb{N}_n$.

Problem 6.3
Figure 6.6 shows 8 distinct points on the circumference of a circle. Assume that no three of the chords formed by these points are concurrent within the circle. Find the number of triangles whose vertices are chosen from the 8 given points or the points of intersection of the chords within the circle.

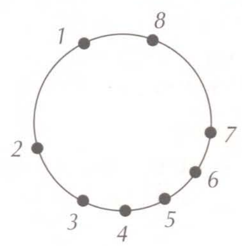


Figure 6.6

Problem 6.4
Find the number of parallelograms contained in the configuration of Figure 6.7 which have no sides parallel to BC.

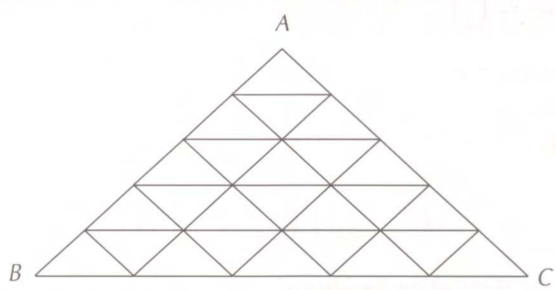


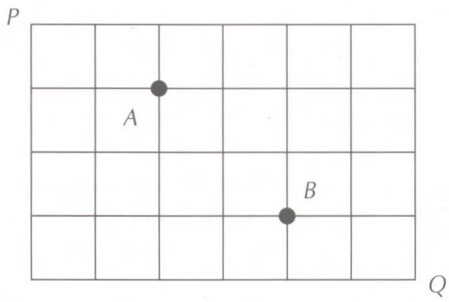
Figure 6.7

Problem 6.5
The number '4' can be expressed as a sum of one or more positive integers, taking order into account, in 8($=2^3$) ways:
$4 = 4 = 1 + 3 = 3 + 1 = 2 + 2 = 1 + 1 + 2 = 1 + 2 + 1$
$= 2 + 1 + 1 = 1 + 1 + 1 + 1.$

Let 'n' be a given natural number. Show, by establishing a bijection between the set of such expressions for n and the set of $(n-1)$-digit binary sequences, that 'n' can be so expressed in 2^{n-1} ways.

Problem 6.6

Find the number of shortest P - Q routes in the following rectangular grid



if (i) the routes must pass through the point A;

(ii) the routes must pass through both points A and B.

7. Distribution of Balls into Boxes

Figure 7.1 shows three distinct boxes into which seven identical balls are to be distributed. Three different ways of distribution are shown in Figure 7.2. (Note that the two vertical bars at the two ends are removed.)

In how many different ways can this be done? This is an example of the type of problems that we shall discuss in this section. We shall see how problems of this type can be solved by applying (BP).

In Figure 7.2, if we treat each vertical bar as a '1' and each ball as a '0', then each way of distribution becomes a 9-digit binary sequence with two 1's. For instance,

(a) —— 0 0 0 0 1 0 0 1 0
(b) —— 0 0 1 0 0 1 0 0 0
(c) —— 0 1 1 0 0 0 0 0 0

Obviously, this correspondence establishes a bijection between the set of ways of distributing the balls and the set of 9-digit binary sequences with two 1's. Thus by (BP), the number of ways of distributing the seven identical balls into three distinct boxes is $\binom{9}{2}$.

Problem 7.1

Show that the number of ways of distributing r identical balls into n distinct boxes is given by

$$\binom{r+n-1}{n-1}\left(=\binom{r+n-1}{r}\right).$$

In the distribution problem discussed above, some boxes are allowed to be vacant. Suppose no box is allowed to be vacant; how many ways are there to distribute the seven identical balls into three distinct boxes?

To meet the requirement that no box is vacant, we first put a ball in each box and this is counted one way as the balls are identical.

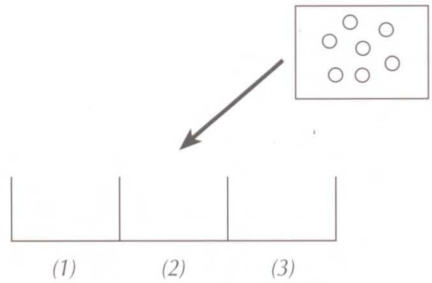


Figure 7.1

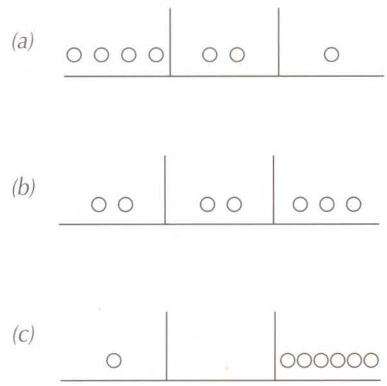


Figure 7.2

We are then left with 4 (= 7 - 3) balls, but we are now free to distribute these 4 balls into any box. By the result of Problem 7.1, the number of ways this can be done is $\binom{4+3-1}{3-1}=\binom{6}{2}$. Thus the number of ways to distribute 7 identical balls into 3 distinct boxes so that no box is empty is $\binom{6}{2}$.

Example 7.1

There are 4 girls and 5 boys in a class, which include 2 particular boys A and B, and a particular girl G. Find the number of ways to arrange all of the boys and girls in a row so that no two of A, B and G are adjacent.

This problem was stated in Problem 5.1 (ix) in [2]. There are different ways to solve the problem. We shall see in what follows that it can be treated as a distribution problem.

First of all, there are 3! ways to arrange A, B and G. Fix one of the ways, say A - B - G. We then consider the remaining six children. Let us imagine tentatively that these 6 children are identical, and they are to be placed in 4 distinct boxes as shown in Figure 7.3 so that boxes (2) and (3) are not vacant (since no two of A, B and G are adjacent). To meet this requirement, we place one in box (2) and one in box (3). Then the remaining four can be placed freely in the boxes in $\binom{4+4-1}{4-1}=\binom{7}{3}$ ways. (Figure 7.4 shows a way of distribution.) But the six children are

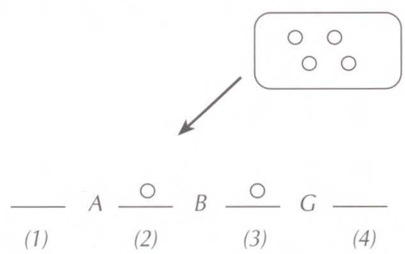


Figure 7.3

actually distinct. Thus, to each of the $\binom{7}{3}$ ways as shown in Figure 7.4, there are 6! ways to arrange them.

Hence by (MP) (see[1]), the required number of ways is $3!\binom{7}{3}\,6!$, which is $6!7\cdot6\cdot5$.

$$\underset{(1)}{\underline{\circ}}\; A \;\underset{(2)}{\underline{\circ\circ}}\; B \;\underset{(3)}{\underline{\circ}}\; G \;\underset{(4)}{\underline{\circ\circ}}$$

Figure 7.4

Remark

The answer $6!7\cdot6\cdot5$ suggests that the problem can be solved in the following way. We first arrange the six children (excluding *A*, *B* and *G*) in a row in 6! ways. Fix one of these ways, say

$$\underset{(1)}{\underline{\quad}}\; X_1 \;\underset{(2)}{\underline{\quad}}\; X_2 \;\underset{(3)}{\underline{\quad}}\; X_3 \;\underset{(4)}{\underline{\quad}}\; X_4 \;\underset{(5)}{\underline{\quad}}\; X_5 \;\underset{(6)}{\underline{\quad}}\; X_6 \;\underset{(7)}{\underline{\quad}}$$

We now consider *A*. There are 7 ways to place *A* in one of the 7 boxes, say box (3):

$$\underset{(1)}{\underline{\quad}}\; X_1 \;\underset{(2)}{\underline{\quad}}\; X_2 \;\underset{(3)}{\underline{\;A\;}}\; X_3 \;\underset{(4)}{\underline{\quad}}\; X_4 \;\underset{(5)}{\underline{\quad}}\; X_5 \;\underset{(6)}{\underline{\quad}}\; X_6 \;\underset{(7)}{\underline{\quad}}$$

Next, consider *B*. Since *A* and *B* cannot be adjacent, *B* can be placed only in one of the remaining 6 boxes. Likewise, *G* can be placed only in one of the remaining 5-boxes. The answer is thus $6!7\cdot6\cdot5$.

Problem 7.2

Show that the number of ways to distribute *r* identical balls into *n* distinct boxes, where $r \geq n$, so that no box is vacant, is given by

$$\binom{r-1}{n-1}\left(=\binom{r-1}{r-n}\right).$$

Problem 7.3

There is a group of 10 students which includes 3 particular students *A*, *B* and *C*. Find the number of ways of arranging the 10 students in a row so that *B* is always between *A* and *C* (*A* and *B*, or *B* and *C* need not be adjacent).

Problem 7.4

Five distinct symbols are transmitted through a communication channel. A total of 18 blanks are to be inserted between the symbols with at least two blanks between every pair of symbols. In how many ways can the blanks be arranged?

8. More Applications of (BP)

We shall give additional examples in this section to show more applications of (BP).

Example 8.1

Consider the following linear equation:

$$x_1 + x_2 + x_3 = 7 \qquad (1)$$

If we put $x_1 = 4$, $x_2 = 1$ and $x_3 = 2$, we see that (1) holds. Since 4, 1, 2 are nonnegative integers, we say that $(x_1, x_2, x_3) = (4, 1, 2)$ is a nonnegative integer solution to the linear equation (1). Note that $(x_1, x_2, x_3) = (1, 2, 4)$ is also a nonnegative integer solution to (1), and so are (1, 2, 4) and (4, 1, 2). Other nonnegative integer solutions to (1) include

$$(0, 0, 7), (0, 7, 0), (1, 6, 0), (5, 1, 1), \ldots$$

How many nonnegative integer solutions to (1) are there?

Let us create 3 distinct 'boxes' to represent x_1, x_2 and x_3 respectively. Then each nonnegative integer solution $(x_1, x_2, x_3) = (a, b, c)$ to (1) corresponds, in a natural way, to a way of distributing 7 identical balls into boxes so that there are *a*, *b* and *c* balls in boxes (1), (2) and (3) respectively (see Figure 8.1). This correspondence clearly establishes a bijection between the set of nonnegative integer solutions to (1) and the set of ways of distributing 7 identical balls in 3 distinct boxes. Thus by (BP) and the result of Problem 7.1, the number of nonnegative integer solutions to (1) is $\binom{7+3-1}{3-1} = \binom{9}{2}$.

$$(4, 1, 2) \text{ —— } \underset{(1)}{\underline{0\,0\,0\,0}} \mid \underset{(2)}{\underline{0}} \mid \underset{(3)}{\underline{0\,0}}$$

$$(2, 5, 0) \text{ —— } \underset{(1)}{\underline{0\,0}} \mid \underset{(2)}{\underline{0\,0\,0\,0\,0}} \mid \underset{(3)}{\underline{\quad}}$$

Figure 8.1

Problem 8.1

Consider the linear equation

$$x_1 + x_2 + \ldots + x_n = r \qquad (2)$$

where *r* is a nonnegative integer. Show that

(i) the number of nonnegative integer solutions to (2) is given by $\binom{r+n-1}{r}$;

(ii)the number of **positive** integer solutions $(x_1, x_2, \ldots, x_n)$, with each $x_i \geq 1$, is given by $\binom{r-1}{r-n}$, where $r \geq n$.

Example 8.2

Recall that the number of 3-element subsets {*a*, *b*, *c*} of the set $\mathbb{N}_{10} = \{1, 2, \ldots, 10\}$ is $\binom{10}{3}$. Suppose now we impose the additional condition that

$$b - a \geq 2 \quad \text{and} \quad c - b \geq 2 \qquad (3)$$

(i.e. no two numbers in {*a*, *b*, *c*} are consecutive). For instance, {1, 3, 8} and {4, 6, 10} satisfy (3) but not {4, 6, 7} and {1, 2, 9}. How many such 3-element subsets of $\mathbb{N}_{10}$ are there?

Let us represent a 3-element subset {*a*, *b*, *c*} of $\mathbb{N}_{10}$ satisfying (3) by a binary sequence as follows:

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
{1, 3, 8}	——	1	0	1	0	0	0	0	1	0	0
{3, 6, 10}	——	0	0	1	0	0	1	0	0	0	1

Note that the rule is similar to the one introduced in Example 6.2. Clearly, this correspondence is a bijection between the set *A* of 3-element subsets of $\mathbb{N}_{10}$ satisfying (3) and the set *B* of 10-digit binary sequences with three 1's in which no two 1's are adjacent. Thus $|A| = |B|$. How to count $|B|$? Using the method discussed in distribution problems, we obtain

$$|B| = \binom{(7-2)+4-1}{4-1} = \binom{8}{3}.$$

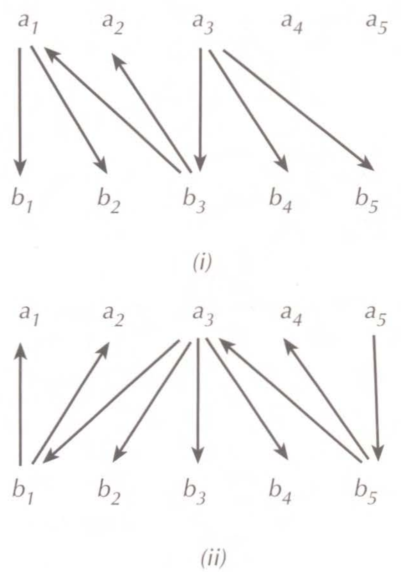
Thus $|A| = \binom{8}{3}$.

Example 8.3

Two tennis teams A and B, consisting of 5 players each, will have a friendly match playing only single tennis with no ties allowed. The players in each team are arranged in order:

$$A: a_1, a_2, a_3, a_4, a_5$$
$$B: b_1, b_2, b_3, b_4, b_5$$

The match is run in the following way. First, a_1 plays against b_1. Suppose a_1 wins (and so b_1 is eliminated). Then a_1 continues to play against b_2. If a_1 wins again, then a_1 keeps on to play against b_3; if a_1 is beaten by b_2 (and so a_1 is eliminated), then b_2 continues to play against a_2, and so on. What is the number of possible ways that the 5 players in team B can be eliminated? Two such ways are shown in Figure 8.2.



'a ⟶ b' means 'a beats b'

Figure 8.2

Let x_i be the number of games won by the player a_i, $i = 1, 2, 3, 4, 5$. Thus, in Figure 8.2 (i),

$$x_1 = 2,\ x_2 = 0,\ x_3 = 3,\ x_4 = x_5 = 0$$

and in Figure 8.2 (ii),

$$x_1 = x_2 = 0,\ x_3 = 4,\ x_4 = 0,\ x_5 = 1.$$

In order for the 5 players in team B to be eliminated, we must have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5, \qquad (4)$$

and the number of ways this can happen is, by (BP), the number of nonnegative integer solutions to equation (4). Thus the desired answer is $\binom{5+5-1}{4} = \binom{9}{4}$.

Problem 8.2

Find the number of 4-element subsets $\{a, b, c, d\}$ of $\mathbb{N}_{20}$ satisfying the following condition

$$b - a \geq 2,\ c - b \geq 3 \ \text{ and } \ d - c \geq 4.$$

Problem 8.3

Find the number of nonnegative integer solutions to the linear equation

$$3x_1 + x_2 + x_3 + x_4 = 12.$$

Problem 8.4

Find the number of triples (x_1, x_2, x_3), where x_1, x_2, x_3 are nonnegative integers, which satisfy the inequality

$$x_1 + x_2 + x_3 \leq 1996.$$

Answers

Problem 6.3

$\binom{8}{3} + 4\binom{8}{4} + 5\binom{8}{5} + \binom{8}{6}$

Problem 6.4

$\binom{7}{4}$

Problem 6.6

(i) $\binom{3}{1}\binom{7}{3}$

(ii) $\binom{3}{1}\binom{4}{2}\binom{3}{1}$

Problem 7.3

$2\binom{10}{3}7!$

Problem 7.4

$5!\binom{13}{3}$

Problem 8.2

$\binom{14}{4}$

Problem 8.3

$\binom{14}{2} + \binom{11}{2} + \binom{8}{2} + \binom{5}{2} + 1$

Problem 8.4

$\binom{1999}{3}$

References

[1] K. M. Koh and B. P. Tan, *Counting - Its Principles and Techniques (1)*, Mathematical Medley Vol 22 March (1995) 8-13.

[2} K. M. Koh and B. P. Tan, *Counting - Its Principles and Techniques (2)*, Mathematical Medley Vol 22 September (1995) 47-51.