Chaotic Dynamical Systems

by Kwek Keng Huat

What is a dynamical system? Basically, any process which evolves with time is an example of a dynamical system. Such systems occur in all branches of science and, indeed, in every aspect of our lives. Weather patterns are examples of huge dynamical systems: the temperature, barometric pressure, wind direction and speed and amount of precipitation are all variables which change with time in this system. The economy is another example of a dynamical system: the rise and fall of the Straits Times Industrials Index is a simple illustration of how the system fluctuates with time. The evolution of the planets in the solar system and simple chemical reactions are examples of other dynamical systems.

The basic goal of studying dynamical systems is to predict the eventual outcome of the evolving process. Namely, if we know in complete detail the past history of a process that is evolving with time, can we deduce the long term behaviour of the system? The answer to this question is sometimes yes and sometimes no. Obviously, prediction of weather systems and stock market fluctuations cannot be made in the long term. On the other hand, we are sure that the sun will rise tomorrow morning and no extraordinary chemical reaction will take place when we add cream to our coffee.

What makes some dynamical systems predictable and others unpredictable? From the above examples, it would seem that dynamical systems which involve a huge number of variables like the weather systems or the economy are unpredictable, whereas systems with fewer variables are easier to understand. However, while this may be true in some cases, it is by no means true in general. Even the simplest of dynamical systems depending on only one variable may yield highly unpredictable and essentially random behaviour. The reason for this is the mathematical notion of chaos.

An Example from Biology

Suppose that there is a single species whose population grows and dwindles over time in a controlled environment. Ecologists have suggested a number of mathematical models to predict the long time behaviour of this population. Here is one of their simplest models.

Suppose that we measure the population of the species at the end of each generation. Let us write P_n for the proportion of population after generation n with $0 \le P_n \le 1$. The model of the growth of this population is

$$P_{n+1} = kP_n(1 - P_n),$$

where k is some constant that depends on ecological conditions such as the amount of food present. This simple quadratic formula is a discrete system with variable P_n. Given P_n and k, we can compute P_{n+1} exactly. In Table 1, we have listed the populations predicted by this model for various values of k. When k is small, the fate of the population seems quite predictable. For $k = 0.5$, the population dies out, whereas for $k = 1.2$, it tends to stablize or reach a definite limiting value. For $k = 3.1$, the limiting values tend to oscillate between two distinct values. For $k = 3.4$, the limiting values oscillate among four values. And finally, for $k = 4$, the initial value $P_0 = 0.5$ leads to disappearance of the species after only two generations; whereas $P_0 = 0.4$ leads to a population count that seems to be completely random.

$P_{n+1} = kP_n(1 - P_n)$					
k					
0.5	1.2	3.1	3.4	4.0	4.0
.5	.5	.5	.5	.4	.5
.125	.3	.775	.85	.96	1
.055	.252	.540	.434	.154	0
.026	.226	.770	.835	.520	0
.013	.210	.549	.469	.998	0
.006	.199	.768	.847	.006	0
.003	.191	.553	.441	.025	0
.002	.186	.766	.838	.099	0
.001	.181	.555	.461	.358	0
.000	.178	.766	.845	.919	0
.000	.176	.556	.446	.298	0
.000	.174	.765	.840	.837	0
.000	.172	.557	.457	.547	0
.000	.171	.765	.844	.991	0
.000	.170	.557	.448	.035	0
.000	.170	.765	.841	.135	0
.000	.169	.557	.455	.466	0
.000	.168	.765	.843	.996	0
.000	.168	.557	.450	.018	0
.000	.168	.765	.851	.070	0
.000	.168	.557	.455	.261	0
.000	.168	.765	.843	.773	0

Table 1

This is the unpredictable nature of this process. Certain *k*-values lead to results which are predictable - a fixed or periodically repeating limiting values. But other *k*-values lead to results which are random.

Let us again consider the system when $k = 4$. We choose three different initial values $P_0^{(1)} = 0.1$, $P_0^{(2)} = 0.10000001$, $P_0^{(3)} = 0.10000002$. The difference between any pair of them is very small. However after the 52nd iteration, there are big differences between three values of P_{52} (See Table 2). Hence a small change in the state of the system at time zero produces a big change in position after a time. In such a case, we say the system is chaotic.

n	$P_{n+1} = 4P_n(1 - P_n)$		
o	$P_0^{(1)} = 0.1$	$P_0^{(2)} = 0.100,000,01$	$P_0^{(3)} = 0.100,000,02$
1	0.36	0.360,000,003,2	0.360,000,006,4
2	0.9216	0.921,600,035,8	0.921,600,071,7
3	0.289,013,76	0.289,013,639,1	0.289,013,518,2
...	... ...		
10	0.147,836,559,9	0.147,824,449,9	0.147,812,518,2
...	... ...		
50	0.277,569,081,0	0.435,657,399,7	0.055,005,377,6
51	0.802,094,386,2	0.983,129,834,6	0.207,919,144,2
52	0.634,955,927,4	0.066,342,251,5	0.658,755,094,6

Table 2

The list of successives, $P_{0,}$ $P_{1,}$ $P_{2,}$ ... of a point P_0 is called the orbit of $P_{0.}$ Generally when a system is iterated on a computer, round-off errors may accumulate and lead to major errors in the predictions. Can we then rely on our computer? Fortunately by a result known as the Shadowing Lemma, there exists a true orbit that is "close" to the pseudo-orbit produced by the computer.

Lorenz butterfly

Edward Lorenz is a meteorologist who worked at the Massachusetts Institute of Technology. As a meteorologist, he was interested in the phenomenon of atmospheric convection. Here is the phenomenon : the sun heats the ground, and therefore the lower layers of atmospheric air become warmer and lighter than that in the higher layers. This causes an upward motion of light, warm air and a downward motion of dense, cold air. These motions consitute convection. Air is a fluid like water, and it should be described by a point in an infinite-dimensional space. By a crude approximation, Lorenz replaced the correct time evolution in an infinite dimension by a time evolution in three dimensions, which he could study on a computer. What came out of the computer is the object shown in the figure, now known as the Lorenz butterfly (Fig. 3). We have to imagine that the point *P* representing the state of our convecting atmosphere is moving with time along the line drawn by the computer. In the situation depicted, the point *P* starts near the origin *O* of the coordinates, then turns around the right "wing" of the butterfly, then a number of times around the left wing, then twice around the right wing, and so on. If the initial position of *P* near *O* was changed just a little bit (so that the difference would not be visible to the naked eye), the details of the figure would be completely changed. The general aspect would remain the same, but the number of successive turns around the right and left wing would be quite different. This is because - as Lorenz recognized - the time evolution of the figure has sensitive dependence on the initial condition. The number of successive turns around the left and right wings is thus erratic, apparently random, and difficult to predict.

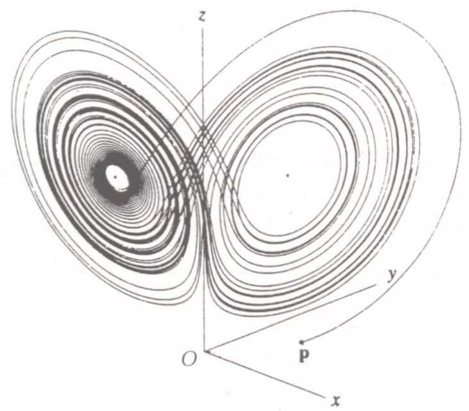
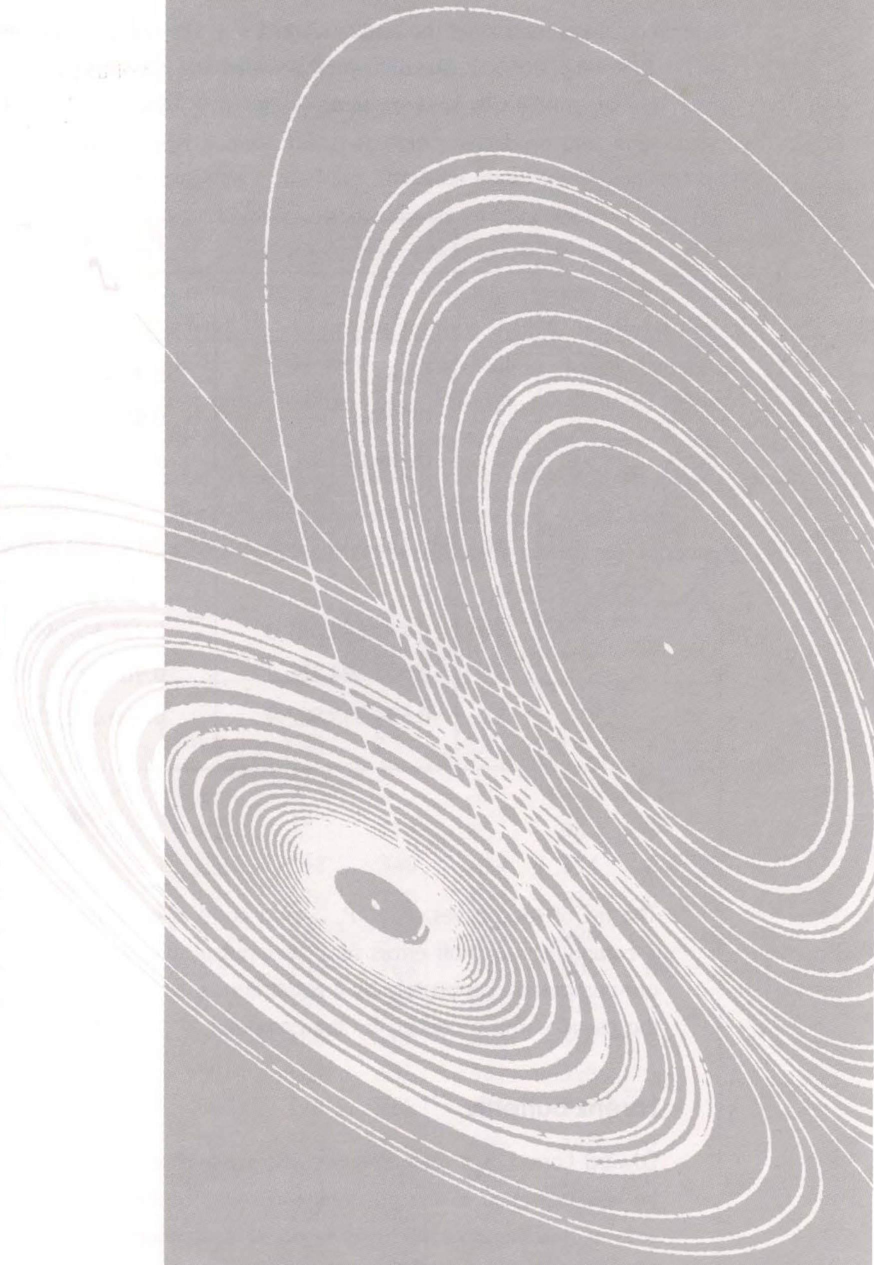


Figure 3

The Lorenz time evolution is not a realistic description of atmospheric convection, but its study nevertheless gave a very strong argument in favour of unpredictability of the motions of the atmosphere. As a meteorologist, Lorenz could thus present a valid excuse for the inability of his profession to produce reliable long-term weather predictions.

Conclusion

Chaos is a feature of natural phenomena which has many applications in engineering, biology and physics. We would like to witness its role, at least qualitatively, in economics, sociology, and the history of mankind. Such disciplines indeed offer problems of greater significance to us than weather predictions. M²



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