Regular and Irregular Sampling

Irene Ong Gek Hiok *
Goh Say Song
Defence Science Organisation
20 Science Park Drive
Singapore 0511

A function in the Paley-Wiener space of bandlimited functions can be reconstructed from certain regularly, or irregularly, sampled values of the function. This paper contains a study of three sampling theorems, namely Shannon's classical sampling theorem, Benedetto and Heller's generalisation of it and Benedetto and Heller's irregular sampling theorem. Experiments have been conducted to compare the performance of the three sampling theorems. For the realisation of the irregular sampling theorem, some examples of irregular sampling sequences that satisfy its hypothesis have been constructed. In addition, a scheme for estimating the frame bounds associated with these sequences, which are required for the implementation of the theorem, has been proposed.

§1. Introduction

Let $L^2(\mathbf{R})$ be the space of complex-valued square-integrable functions over the real line. The norm of a function f in $L^2(\mathbf{R})$ is given by

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2},$$

which is finite. Also, for any $f, g \in L^2(\mathbf{R})$, define an inner product $\langle \, , \, \rangle$ by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt.$$

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* Year III student in Computational Science, National University of Singapore, on industrial attachment to Defence Science Organisation.

Formally, we define the *Fourier transform* $\hat{f}$ of a function f to be

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\gamma}dt.$$

The *Fourier inversion formula* is given by

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\gamma)e^{2\pi it\gamma}d\gamma;$$

and the *convolution* of two functions f and g over the real line is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

For any $\Omega > 0$, define the *Paley-Wiener space* PW_Ω of *finite energy, bandlimited functions* to be

$$PW_\Omega = \{f \in L^2(\mathbf{R}) : \text{supp}\hat{f} \subseteq [-\Omega, \Omega]\}.$$

The leading example of a function in PW_Ω is the *sinc function*

$$k(t) = \text{sinc}(2\pi\Omega t) = \sin(2\pi\Omega t)/(2\pi\Omega t). \quad (1)$$

With respect to the sinc function, it is known (see [1]) that PW_Ω is a *reproducing kernel Hilbert space*, that is,

$$PW_\Omega = \{f \in L^2(\mathbf{R}) : 2\Omega(f * k) = f\}. \quad (2)$$

Other examples of functions in PW_Ω are functions of the form

$$f(t) = [\text{sinc}(2\pi\omega t)]^m, \quad (3)$$

where m is a positive integer, and $\omega \leq \Omega/m$. (Note that if f is of the form (3), then supp $\hat{f} = [-\omega m, \omega m]$.) In addition, linear combinations, as well as translations along the real line, of such functions are also in PW_Ω.

In this paper, we shall study the reconstruction of any function in PW_Ω, based on certain regularly, or irregularly, sampled values of the function. Two regular sampling theorems will be examined in Section 2. A characterisation of functions in PW_Ω with respect to their real and imaginary parts will also be obtained. In Section 3, we shall study an irregular sampling theorem, and construct sampling sequences that satisfy its hypothesis. Furthermore, a scheme for estimating the frame bounds associated with these seqences, which are required for the implementation of the theorem, will be proposed. Finally, in Section 4, we shall show the results of some numerical experiments which compare the performance of the above sampling theorems.

§2. Regular Sampling

In this section, we shall reconstruct a given function in the Paley-Wiener space PW_Ω based on regular samples of the function, that is, sampled values are taken at a sequence of uniformly spaced points. Two regular sampling theorems, where the second is an extension of the first, will be examined.

The first theorem which we shall consider is the classical sampling theorem.

Theorem 1. **(Shannon [3])** *Let $T, \Omega > 0$. If $f \in PW_\Omega$ and $0 < 2T\Omega \leq 1$, then*

$$f(t) = 2T\Omega \sum_{n=-\infty}^{\infty} f(nT)\text{sinc}[2\pi\Omega(t - nT)], \tag{4}$$

where the convergence is uniform on $\mathbf{R}$ and in $L^2(\mathbf{R})$-norm.

Remarks.

(1) The formula (4) expresses f in terms of its sampled values $\{f(nT) : n \in \mathbf{Z}\}$, with $\{nT : n \in \mathbf{Z}\}$ as the sampling sequence. Here, T is the *sampling period*, and the reciprocal of T gives the *sampling frequency*, or the *sampling rate.*

(2) The sampling rate, 2Ω, which occurs when $2T\Omega = 1$, is known as the *Nyquist rate.* It is the minimum sampling rate required for perfect reconstruction of functions in PW_Ω based on their sampled values. Below this rate, perfect reconstruction is generally impossible, due to the phenomenon called *aliasing* (see [1]).

(3) It is known (see [1]) that if (4) holds for all $f \in PW_\Omega$, then $2T\Omega \leq 1$.

The second regular sampling theorem which we shall examine is

Theorem 2. **(Benedetto and Heller [2])** *Let $T, \Omega > 0$ with $0 < 2T\Omega \leq 1$. Suppose that $s \in PW_{1/(2T)}$ satisfies the condition $\hat{s} = T$ on $[-\Omega, \Omega]$. Then for every $f \in PW_{\Omega'}$*

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)s(t - nT), \tag{5}$$

where the convergence is uniform on $\mathbf{R}$ and in $L^2(\mathbf{R})$-norm.

Remarks.

(1) The function s is known as the *kernel* or the *sampling function.*

(2) If we define

$$\hat{s}(\gamma) = \begin{cases} T, & \text{if } \gamma \in [-\Omega, \Omega]; \\ 0, & \text{otherwise;} \end{cases}$$

then

$$s(t) = 2T\Omega \operatorname{sinc}(2\pi\Omega t),$$

and (5) reduces to (4).

In Theorem 2, the functions f and s are generally complex-valued. Let us write

$$f(t) = g(t) + ih(t),$$

and

$$s(t) = u(t) + iv(t),$$

where g, h, u and v are real-valued functions. Then it is immediate from (5) that

$$g(t) + ih(t) = \sum_{n=-\infty}^{\infty} [g(nT) + ih(nT)][u(t - nT) + iv(t - nT)].$$

Equating real and imaginary parts, we obtain

$$g(t) = \sum_{n=-\infty}^{\infty} [g(nT)u(t - nT) - h(nT)v(t - nT)], \tag{6}$$

and

$$h(t) = \sum_{n=-\infty}^{\infty} [g(nT)v(t - nT) + h(nT)u(t - nT)]. \tag{7}$$

Discussion. If f is *real-valued*, then (6) and (7) yield

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)u(t - nT), \tag{8}$$

and

$$0 = \sum_{n=-\infty}^{\infty} f(nT)v(t - nT). \tag{9}$$

(1) Equation (8) indicates that the function f can be reconstructed from its sampled values, using only the real part of s. Based on this observation, we may show that a function $r \in PW_{1/(2T)}$ which can be used as the kernel for reconstructing real-valued functions in PW_Ω need not satisfy $\hat{r} = T$ on $[-\Omega, \Omega]$. Indeed, consider a function $s \in PW_{1/(2T)}$ such that

$$\hat{s}(\gamma) = \begin{cases} 2T^2(\gamma - 1/(2T))/(2T\Omega - 1), & \text{if } \gamma \in (\Omega, 1/(2T)]; \\ T, & \text{if } \gamma \in [-\Omega, \Omega]; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Then taking the inverse Fourier transform of $\hat{s}$, and defining r to be $Re\{s\}$, we obtain

$$r(t) = T\Omega\text{sinc}(2\pi\Omega t) - [\text{sinc}^2(\pi t/(2T))/4 - T^2\Omega^2\text{sinc}^2(\pi\Omega t)]/(2T\Omega - 1). \tag{11}$$

It follows from (8) that r can be used as a kernel to reconstruct any real-valued function f in PW_Ω from its sampled values. However, a direct calculation gives

$$\hat{r}(\gamma) = \begin{cases} 2T^2(\gamma - 1/(2T))/(2T\Omega - 1), & \text{if } \gamma \in (\Omega, 1/(2T)]; \\ 3T/2, & \text{if } \gamma \in [-\Omega, \Omega]; \\ -2T^2(\gamma + 1/(2T))/(2T\Omega - 1), & \text{if } \gamma \in [-1/(2T), -\Omega); \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\hat{r} \not\equiv T$ on $[-\Omega, \Omega]$.

(2) It is interesting to observe that for a fixed v, where v is the imaginary part of a function s satisfying the hypothesis of Theorem 2, *any* function f in PW_Ω satisfies (9).

We shall now give some examples of real-valued functions f in PW_Ω. For the construction of such functions, the following proposition is often useful.

Proposition 1. A complex-valued function f is in PW_Ω if and only if

$$f = g + ih$$

for some real-valued functions g and h in PW_Ω.

Proof. Suppose that f is in PW_Ω. Since PW_Ω is characterised by (2) (see Section 1), by taking real and imaginary parts, it follows that

$$2\Omega(g * k) = g,$$

and

$$2\Omega(h * k) = h,$$

where k is given by (1), $g = Re\{f\}$ and $h = Im\{f\}$. Observe that

$$|Re\{f\}|, |Im\{f\}| \leq |f|.$$

Since f is in $L^2(\mathbf{R})$, g and h are also in $L^2(\mathbf{R})$. Thus (2) implies that g, $h \in PW_\Omega$.

Conversely, if $f = g + ih$ for some real-valued functions g and h in PW_Ω, then

$$|f|^2 = g^2 + h^2;$$

whence $f \in L^2(\mathbf{R})$. Using similar arguments as above, it follows from the characterisation (2) that $f \in PW_\Omega$, and the proof of Proposition 1 is complete.

Remarks.

(1) Proposition 1 provides us with a scheme for constructing real-valued functions in PW_Ω. Indeed, consider a function $F \in L^2(\mathbf{R})$ with supp $F \subseteq [-\Omega, \Omega]$. By taking the inverse Fourier transform of F, we obtain the function $f = g + ih$, where g and h are real-vaued functions. Since f is in PW_Ω, it follows from Proposition 1 that g and h are real-valued functions in PW_Ω.

(2) It is sufficient to perform numerical experiments on Theorems 1 and 2 with only real-valued functions. Extension of experimental results to complex-valued functions in PW_Ω is simple, since Proposition 1 characterises PW_Ω in terms of real-valued functions in the space.

Some examples of real-valued functions f in PW_Ω are listed in Table 1. Note that the function f_3 is obtained by defining it to be the imaginary part of the inverse Fourier transform of the function

$$F(\gamma) = \begin{cases} 1, & \text{if } \gamma \in [0, \Omega]; \\ -1, & \text{if } \gamma \in [-\Omega, 0); \\ 0, & \text{otherwise.} \end{cases}$$

i	f_i
1	$\mathrm{sinc}^2(\pi\Omega t)$
2	$2\mathrm{sinc}^2(\pi\Omega(t-1)) - 5\mathrm{sinc}^3(0.6\pi\Omega(t+5)) + \mathrm{sinc}^7(0.2\pi\Omega(t-3))$
3	$(1-\cos(2\pi\Omega t))/(\pi t)$

Table 1. Examples of functions f in PW_Ω.

Since we are considering only real-valued functions f, by (8), it suffices to use kernels that are the real parts of functions s which satisfy the hypothesis of Theorem 2. Some examples of such kernels are stated in Table 2. The right-hand column of Table 2 gives the rates of decay of the kernels $s_j(t)$ as t tends to $\pm\infty$. Note that the functions s_1 and s_2 are obtained by taking the inverse Fourier transform of the functions S_1 and S_2 respectively, where

$$S_1(\gamma) = \begin{cases} T, & \text{if } \gamma \in [-\Omega, \Omega]; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$S_2(\gamma) = \begin{cases} 2T^2(\gamma - 1/(2T))/(2T\Omega - 1), & \text{if } \gamma \in (\Omega, 1/(2T)]; \\ T, & \text{if } \gamma \in [-\Omega, \Omega]; \\ -2T^2(\gamma + 1/(2T))/(2T\Omega - 1), & \text{if } \gamma \in [-1/(2T), -\Omega); \\ 0, & \text{otherwise.} \end{cases}$$

The function s_3 is obtained by defining it to be the real part of the inverse Fourier transform of the function

$$S_3(\gamma) = \begin{cases} e^\gamma, & \text{if } \gamma \in [\Omega, 1/(2T)]; \\ T, & \text{if } \gamma \in [-\Omega, \Omega]; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, s_4 is precisely the function r in (11), which is the real part of the inverse Fourier transform of the function in (10).

j	s_j	Order
1	$2T\Omega\mathrm{sinc}(2\pi\Omega t)$	$O(t^{-1})$
2	$T^2(\cos(2\pi\Omega t) - \cos(\pi t/T))/(\pi^2 t^2(1-2T\Omega)^2)$	$O(t^{-2})$
3	$2T\Omega\mathrm{sinc}(2\pi\Omega t) + \{e^{1/(2T)}\cos(\pi t/T) - e^\Omega\cos(2\pi\Omega t) +$ $2\pi t(e^{1/(2T)}\sin(\pi t/T) - e^\Omega\sin(2\pi\Omega t))\}/(1+4\pi^2t^2)$	$O(t^{-1})$
4	$T\Omega\mathrm{sinc}(2\pi\Omega t) - (\mathrm{sinc}^2(\pi t/(2T))/4 - T^2\Omega^2\mathrm{sinc}^2(\pi\Omega t))/(2T\Omega - 1)$	$O(t^{-1})$

Table 2. Examples of kernels s.

Note that for our numerical experiments in Section 4, we shall use the functions f and s given in Tables 1 and 2.

§3. Irregular Sampling

We shall now introduce the notions of a frame and a frame operator, which will be used in the formulation of the irregular sampling theorem.

Definitions.

(1) Let H be a separable Hilbert space. A sequence $\{h_n\} \subseteq H$ is a *frame* if there exist constants $A, B > 0$ such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_{n=-\infty}^{\infty} |\langle f, h_n\rangle|^2 \leq B\|f\|^2,$$

where $\langle \ , \ \rangle$ is the inner product on H and $\|f\| \equiv \langle f, f\rangle^{1/2}$ is the norm of f in H. The constants A and B are known as *frame bounds.*

(2) The *frame operator* of the frame $\{h_n\}$ is the function $S : H \to H$ defined by

$$Sf = \sum_{n=-\infty}^{\infty} \langle f, h_n\rangle h_n.$$

(3) Let $L^2[-\Omega_1, \Omega_1]$ be the separable Hilbert space of square-integrable functions over $[-\Omega_1, \Omega_1]$. Define $e_t(\gamma) = e^{2\pi i t\gamma}$. Then a sequence $\{e_{t_n}\}$ is said to be a *Fourier frame* for $L^2[-\Omega_1, \Omega_1]$ if it is a frame for $L^2[-\Omega_1, \Omega_1]$.

The irregular sampling theorem that we shall examine is

Theorem 3. (Benedetto and Heller [2]) *Suppose that* $0 < \Omega < \Omega_1$. *Let the sequence* $\{t_n\} \subseteq \mathbf{R}$ *have the property that* $\{e_{-t_n}\}$ *is a Fourier frame for* $L^2[-\Omega_1, \Omega_1]$ *with frame operator* S. *Also, suppose that* s *satisfies* $\hat{s} \in L^\infty(\mathbf{R})$, *supp* $\hat{s} \subseteq [-\Omega_1, \Omega_1]$ *and* $\hat{s} = 1$ *on* $[-\Omega, \Omega]$. *Then for every* $f \in PW_\Omega$,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f) s(t - t_n), \qquad (12)$$

where the convergence is in $L^2(\mathbf{R})$*-norm, and*

$$c_n(f) = \langle S^{-1}\hat{f}, e_{-t_n}\rangle_{[-\Omega_1,\Omega_1]}.$$

Remark. It can be shown (see [1]) that for $f \in PW_\Omega$ and $\Omega < \Omega_1$,

$$c_n(f) = 2/(A+B)\sum_{k=0}^{\infty}\langle (I - 2S/(A+B))^k \hat{f}, e_{-t_n}\rangle_{[-\Omega_1,\Omega_1]}, \tag{13}$$

where A and B are the frame bounds. Furthermore, by approximating $c_n(f)$ with the first two terms of the expansion in (13), we obtain

$$c_n(f) \cong [4f(t_n)/(A+B)] - [8\Omega_1/(A+B)^2][f(t_n) + \sum_{m\neq n} f(t_m)\mathrm{sinc}(2\pi\Omega_1 t)]. \tag{14}$$

We shall now construct sampling sequences $\{t_n\}$ that satisfy the hypothesis of Theorem 3. The following theorem is particularly useful for our construction.

Theorem 4. (Benedetto [1]) *Suppose that* $\Omega_1 > 0$. *Let the sequence* $\{t_n\} \subseteq \mathbf{R}$ *have the property that* $\{t_n\}$ *is strictly decreasing,* $\lim_{n\to\pm\infty} t_n = \mp\infty$, *and*

$$0 < d \leq \inf(t_n - t_{n+1}) \leq \sup(t_n - t_{n+1}) = T < \infty.$$

Assume that $2T\Omega_1 < 1$. *Then* $\{e_{-t_n}\}$ *is a Fourier frame for* $L^2[-\Omega_1,\Omega_1]$ *with frame bounds* A *and* B *satisfying*

$$(1-2T\Omega_1)^2/T \leq A \leq B \leq 4(e^{\pi\Omega_1 d} - 1)/(\pi^2\Omega_1 d^2). \tag{15}$$

Remark. It is immediate from (15) that

$$B/A \leq 4T(e^{\pi\Omega_1 d} - 1)/(\pi^2\Omega_1 d^2(1-2T\Omega_1)^2). \tag{16}$$

Construction of Irregular Sampling Sequences

(1) For $0 < d \leq T < 1/(2\Omega_1)$, suppose that

$$t_n = -(dn + (T-d)\varepsilon_n/M), \tag{17}$$

where $n \in \mathbf{Z}$, and $\{\varepsilon_n\}$ is a nonconstant increasing sequence with $M = \sup(\varepsilon_{n+1} - \varepsilon_n)$ finite. Such a sequence $\{t_n\}$ satisfies Theorems 3 and 4. If, in addition, the sequence $\{\varepsilon_n\}$ satisfies

$$(\varepsilon_{n+1} - \varepsilon_n) \leq (\varepsilon_n - \varepsilon_{n-1}), \tag{18}$$

for $n \geq 1$; and

$$(\varepsilon_{n+1} - \varepsilon_n) \geq (\varepsilon_n - \varepsilon_{n-1}), \tag{19}$$

for $n \leq -1$, then $M = \max\{(\varepsilon_1 - \varepsilon_0), (\varepsilon_0 - \varepsilon_{-1})\}$. Indeed, (18) implies that

$$(\varepsilon_1 - \varepsilon_0) \geq (\varepsilon_2 - \varepsilon_1) \geq ...;$$

whence $\sup\{(\varepsilon_{n+1} - \varepsilon_n) : n \geq 1\} = (\varepsilon_1 - \varepsilon_0)$. Similarly, (19) gives $\sup\{(\varepsilon_{n+1} - \varepsilon_n) : n \geq -1\} = (\varepsilon_0 - \varepsilon_{-1})$.

It should be noted that if $\varepsilon_n = \phi(n)$ for every $n \in \mathbf{Z}$, where $\phi : \mathbf{R} \to \mathbf{R}$ is increasing, concave on $(0, \infty)$ and convex on $(-\infty, 0)$, then (18) and (19) are satisfied. For instance, $\varepsilon_n = \tan^{-1} n$. In this case, $M = \max\{(\tan^{-1} 1 - \tan^{-1} 0), (\tan^{-1} 0 - \tan^{-1}(-1))\} = \pi/4$. Consequently, (17) reduces to

$$t_n = -(dn + 4(T-d)\tan^{-1} n/\pi). \tag{20}$$

In the numerical experiments of Section 4, we shall use the sampling sequence defined by (20).

(2) The Fourier frame associated with the sequence $\{t_n\}$ in (17) has frame bounds A and B satisfying (16). If $(B/A - 1)$ is sufficiently small, (14) is a good approximation of (13). Thus we seek d and T that give sufficiently small $(B/A - 1)$. For $0 < d \leq T < 1/(2\Omega_1)$, define

$$X = 4T(e^{\pi\Omega_1 d} - 1)/(\pi^2 \Omega_1 d^2 (1 - 2T\Omega_1)^2) - 1.$$

Then $(B/A - 1) \leq X$, and when X is sufficiently small, $(B/A - 1)$ is sufficiently small. Table 3 shows the values of X, A_1 and B_1 that correspond to some values of d and T for $\Omega_1 = 1$; and A_1 and B_1

are the lower bound for A and upper bound for B (rounded to the nearest integer) respectively.

d	T	X	$A_1 = (1-2T\Omega_1)^2/T$	$B_1 = 4(e^{\pi\Omega_1 d}-1)/(\pi^2\Omega_1 d^2)$
0.015	0.017	0.583	54	87
0.028	0.030	0.614	29	48
0.031	0.032	0.576	27	44
0.033	0.041	0.978	20	41
0.046	0.047	0.705	17	30
0.053	0.059	0.982	13	27

Table 3. Values of X, A_1 and B_1 that correspond to some values of d and T for $\Omega_1 = 1$.

(3) It is sufficient to compute Table 3 for $\Omega_1 = 1$. Indeed, for any $\Omega_1 > 0$, define $d' = d/\Omega_1$ and $T' = T/\Omega_1$. Then

$$X = 4T'(e^{\pi\Omega_1 d'}-1)/(\pi^2\Omega_1(d')^2(1-2T'\Omega_1)^2)-1$$
$$= 4T(e^{\pi d}-1)/(\pi^2 d^2(1-2T)^2)-1.$$

Thus, if d and T give sufficiently small $(B/A-1)$ when $\Omega_1 = 1$, then d' and T' defined above also produce the same result for an arbitrary $\Omega_1 > 0$, and

$$t_n = -(d'n + (T'-d')\varepsilon_n/M)$$

can be used as a sampling sequence. For instance, if $\Omega_1 = 0.6$, $d = 0.033$ and $T = 0.041$, then $d' = 0.0550$ and $T' = 0.0683$. We shall use this pair of d' and T' in Section 4.

(4) To find $c_n(f)$ for a sampling sequence defined above, we need an approximation of the constant $(A+B)$ (see (14)). By construction, $A_1 \le A \le B \le B_1$, where A_1 and B_1 are defined as in Table 4. Since the frame bounds A and B are independent of the function f in PW_Ω, $(A+B)$ may be taken to be the value of (A_0+B_0) that gives the best possible reconstruction of a particular function f using (12) among all integers A_0's and B_0's satisfying $A_1 \le A_0 \le B_0 \le B_1$. An example of such a function is $f(t) = \text{sinc}^2(\pi\Omega t)$.

§4. Numerical Experiments

In this section, we shall perform some numerical experiments on Theorems 2 and 3.

§4.1. Regular sampling

In our experiments on Theorem 2, we shall approximate the functions f_i, for $i = 1, 2, 3$ (see Section 2), by $f_{i,j,N}$, where Ω is taken to be 0.5, and $f_{i,j,N}$ is defined by

$$f_{i,j,N}(t) = \sum_{n=-N}^{N} f_i(nT)s_j(t - nT). \tag{21}$$

Here, N is a positive integer, t runs from -10 to 10 while stepping up by 0.1, and s_j, for $j = 1, ..., 4$, are the kernels given in Section 2. All the four kernels are even functions and they tend to zero as t tends to $\pm\infty$.

Our aim here is to compare the performance of the different kernels at various sampling rates. Let $N(T, R)$ denote the minimum N required to attain a signal-to-noise ratio of R dB when the sampling period is T. Tables 4 to 6 show the values of $N(T, R)$ for the functions f_1, f_2 and f_3.

s_j	$N(.1, 30)$	$N(.1, 50)$	$N(.1, 70)$	$N(.99, 30)$	$N(.99, 50)$	$N(.99, 70)$
1	29	89	113	3	9	15
2	32	91	97	4	10	14
3	60	114	212	4	10	14
4	29	89	108	3	9	15

Table 4. Regular sampling experiments on f_1.

s_j	$N(.1, 30)$	$N(.1, 50)$	$N(.1, 70)$	$N(.99, 30)$	$N(.99, 50)$	$N(.99, 70)$
1	69	97	105	7	10	10
2	71	98	102	8	11	11
3	60	114	212	4	10	14
4	29	89	108	3	9	15

Table 5. Regular sampling experiments on f_2.

s_j	$N(.1, 30)$	$N(.1, 50)$	$N(.1, 70)$	$N(.99, 30)$	$N(.99, 50)$	$N(.99, 70)$
1	112	> 1000	> 1000	13	104	> 1000
2	94	98	108	14	28	36
3	750	> 1000	> 1000	14	26	30
4	92	512	>1000	13	35	510

Table 6. Regular sampling experiments on f_3.

Note that perfect reconstruction of a function is generally not possible if we undersample, that is, sample below Nyquist rate. If we undersample, it is not possible to reconstruct any of the f_i's with $N \leq 1000$, while atttaining a signal-to-noise ratio of 30 dB. (In numerical examples, a signal-to-noise ratio of 30 dB can be easily attained if we sample at, or above, Nyquist rate.) This is due to the phenomenon of aliasing. On the other hand, good approximations of the f_i's are easily obtained while sampling at, or above, Nyquist rate. Indeed, a signal-to-noise ratio of 70 dB can be attained with N much less than 1000. If we sample slightly above Nyquist rate, all the four kernels give similar results. This is because $\hat{s}_1, ..., \hat{s}_4$ differ only in the interval $[-1/(2T), -\Omega) \cup (\Omega, 1/(2T)]$, which is small when the sampling rate is close to that of Nyquist.

If we oversample, that is, sample above Nyquist rate, then more terms have to be taken before good approximations are possible, as (5) is essentially an interpolation formula. Thus when T is small, N has to be large. Observe that while oversampling, s_2 performs better than the other three kernels, in the sense that less terms need to be taken for good approximations of the f_i's. This is because as t tends to $\pm\infty$, s_2 decays in $O(t^{-2})$, whereas the other kernels decay in $O(t^{-1})$ (see Table 2). Note that when n is large, a term in the series (5) is small, and thus can be dropped. Since s_2 decays faster than the other kernels, less terms are needed to obtain a good approximation of the infinite series (5).

Finally, note that we have also performed the same numerical experiments with $\Omega = 7$, and similar results were obtained.

§4.2. Irregular sampling

In our experiments on Theorem 3, we shall assume that $\Omega = 0.5$ and $\Omega_1 = 0.6$, and use the sampling sequence

$$t_n = -(d'n + 4(T' - d')\tan^{-1} n/\pi),$$

where $d' = 0.0550$ and $T' = 0.0683$. We shall attempt to reconstruct the same functions f_1, f_2 and f_3 as in Section 4.1. The kernels used here are s_5 and s_6, which are given by

$$s_5(t) = 2\Omega \text{sinc}(2\pi\Omega t),$$

and

$$s_6(t) = (\cos(2\pi\Omega t) - \cos(2\pi\Omega_1 t))/(2\pi^2 t^2(\Omega_1 - \Omega)).$$

These kernels are obtained by taking the inverse Fourier transform of

$$S_5(\gamma) = \begin{cases} 1, & \text{if } \gamma \in [-\Omega, \Omega]; \\ 0, & \text{otherwise}; \end{cases}$$

and

$$S_6(\gamma) = \begin{cases} (\gamma - \Omega_1)/(\Omega - \Omega_1), & \text{if } \gamma \in (\Omega, \Omega_1]; \\ 1, & \text{if } \gamma \in [-\Omega, \Omega]; \\ -(\gamma + \Omega_1)/(\Omega - \Omega_1), & \text{if } \gamma \in [-\Omega_1, -\Omega); \\ 0, & \text{otherwise}; \end{cases}$$

respectively. The functions f_i are approximated by

$$\tilde{f}_{i,j}(t) = \sum_{n=-1000}^{1000} c_n(f_i) s_j(t - t_n),$$

where $c_n(f_i)$ is given by (14), t runs from -10 to 10 while stepping up by 0.1, and s_j, for $j = 5, 6$, are the kernels given above. The signal-to-noise ratio obtained by this approximation is compared to that obtained by using (21) with $N = 1000$ and $j = 1, 2$. Three sampling periods for (21) are considered. They are the smallest (d'), average (av) and largest (T') difference between consecutive sampling points from $n = -1000$ to $n = 1000$ in the irregular sampling sequence $\{t_n\}$. The results of the experiments are shown in Table 7, where SNR(s_j, P) denotes the signal-to-noise ratio obtained when using the kernel s_j. If the regular sampling formula (21) is used, then P denotes the sampling period.

f_i	Irreg. Samp.		Reg. Samp.					
	SNR $(s_5, -)$	SNR $(s_6, -)$	SNR (s_1, d')	SNR (s_1, av)	SNR (s_1, T')	SNR (s_2, d')	SNR (s_2, av)	SNR (s_2, T')
1	61.9	61.6	123.3	123.3	129.6	163.5	163.5	167.0
2	56.6	56.6	113.0	113.0	119.1	173.9	173.9	177.8
3	44.2	57.8	44.5	44.6	46.4	136.7	136.7	140.3

Table 7. Comparison of experimental results between regular and irregular sampling.

We observe that although fairly good approximations are obtained using irregular sampling, regular sampling generally gives even better approximations. We have also performed the same numerical experiments with $\Omega = 7$, and similar results are obtained.

§5. Conclusion

We have studied the reconstruction of a function in the Paley-Wiener space from certain regularly, or irregularly, sampled values of the function. Among the cases we have examined, the regular sampling theorems (Theorems 1 and 2) give better results than the irregular sampling theorem (Theorem 3). In our experiments, the average sampling rate for irregular sampling is much higher than Nyquist rate, the minimum regular sampling rate required for perfect reconstruction. However, for the same number of terms taken, the regular sampling theorems approximate a given function better than the irregular sampling theorem. It is not known to us whether there are situations in which the reverse is true.

Another outstanding issue is to find a better way of approximating the frame bounds A and B that correspond to the irregular sampling sequence $\{t_n\}$. Our method provides only a rough estimate of $(A+B)$, although it gives satisfactory experimental results.

Acknowledgements.

The authors would like to thank Mr Quek Gim Pew, Division Head (Sensor Research) of Defence Science Organisation, for his support, and Dr Antony Ng and Mr Yan Chye Hwang, also of Defence Science Organisation, for their assistance.

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