Computer Graphics and Soap Bubbles

Cheung Leung Fu & Ng Tze Beng
Department of Mathematics
National University of Singapore
Singapore 0511

Soap Bubbles

From years of study and of contemplation
An old man brews a work of clarity,
A gay and involuted dissertation
Discoursing on sweet wisdom playfully

An eager student bent on storming heights
Has delved in archives and in libraries,
But adds the touch of genius when he writes
A first book full of deepest subtleties.

A boy, with bowl and straw, sits and blows,
Filling with breath the bubbles from the bowl.
Each praises like a hymn, and each one glows;
Into the filmy beads he blows his soul.

Old man, student, boy, all these three
Out of the Maya-foam of the universe
Create illusions. None is better or worse
But in each of them the Light of Eternity
Sees its reflection, and burns more joyfully.

The Glass Bead Game
Hermann Hesse

The soap bubble, a wonderful toy for children, a motif for poetry, is at the same time a mathematical entity that has a fairly long history. Thanks to the advancement of computer graphics which led to solutions of decade-old conjectures, 'soap bubbles' have become once again a focus of much mathematical research in recent years.

Anyone who has seen or played with a soap bubble knows that it is a spherical object. It is indeed a sphere if one defines a *soap bubble* to be a surface minimizing the area among all surfaces enclosing the same given volume. Relaxing this condition, mathematicians defined a *soap bubble* or *surface of constant mean curvature* to be a surface which is *locally* the stationary point of the area function when compared again *locally* to neighbouring surfaces enclosing the same volume. A mathematician's soap bubble can behave strangely: neither is it usually spherical, nor does it need to enclose a finite volume. In fact, it can even intersect itself wildly, be infinitely long or have a lot of *holes*!

Having defined a mathematical notion, mathematicians began to look for examples. To meet this end, the definition above seems to be too abstract and hence of little help. Fortunately, these *soap bubbles* can be characterized by a rather simple entity called *mean curvature.* Now, let us explain what a *mean curvature* is. Imagine a *soap bubble* sitting in our three dimensional Euclidean space. One can readily imagine that, at every point on the *soap bubble*, which is naturally a surface, there is a normal direction defined. Consider any plane having this normal direction (there are infinitely many of them!) as an axis; we can easily see that each of them intersects the surface in a plane curve. Advanced Calculus tells us that to each such curve is associated an entity called curvature. Since there are now infinitely many planes containing the same normal, there are infinitely many intersection curves and hence infinitely many such curvatures associated with them. Mathematicians define the average of the largest such curvature and the smallest one as the *mean curvature.* Curiously enough, a *soap bubble* is a surface with constant non-zero *mean curvature.*

It was the French mathematician Delaunay who first discovered a method of finding a lot of such *soap bubbles.* He was interested in finding cylindrically symmetric *soap bubbles* and found that they can all be generated by rotating some quadrics, without slipping about a straight line. The cylinder is the simplest such soap bubble. Much later, in the seventies, H. B. Lawson [4] discovered another method of generating a family of *soap bubbles* from a given one. His method consists in turning an entity on a *soap bubble* called the *second fundamental form* by an arbitrary angle between 0° and 360°. He discovered that this 'internal rotation' of the *soap bubble* generates new ones which are in fact 'isometric' to the original one, i.e. they have the same 'length' and 'angle' on them. To give this family of *soap bubbles* a name, he called it the family of associated surfaces to the original one. The following pictures, which we generated by using Maple V based on the theoretical work of Dajezer and Do Carmo [2] and the Mathematica pictures of Hitt and Roussos [3], show us how the associated surfaces differ from the original one. A typical phenomenon is that the cylindrical symmetry is destroyed. Hence a cyclindrical *soap bubble* evolves into a family of helicoidal *soap bubbles* and the cylindrical symmetry is recovered only when we reach the angle 180°.

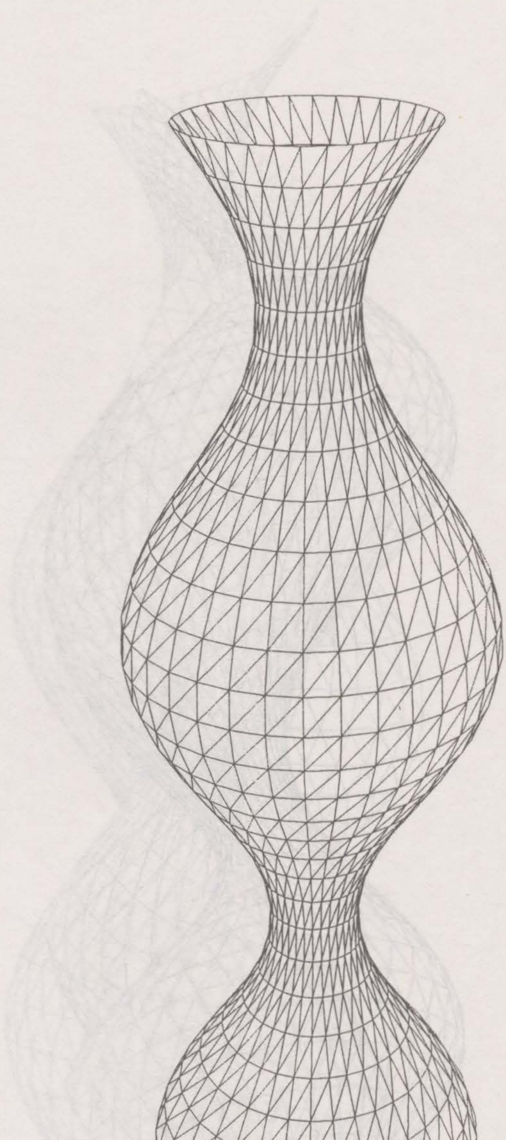
Open Problem

Our pictures and the corresponding ones of Hitt and Roussos [3] (see also [2]) suggest that there are *soap bubbles* which are topologically the same as a plane (the paper of Hitt and Roussos [3] in fact contains a proof). This is also our original motivation for looking at these pictures.

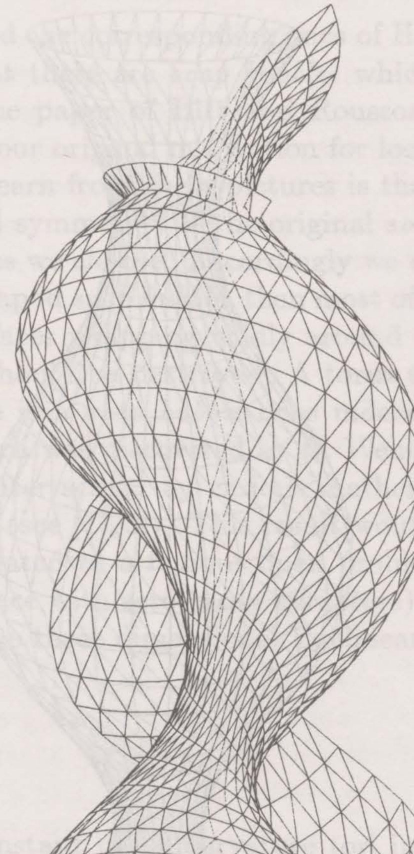
The lesson we learn from these pictures is that some symmetries (in this case: cylindrical symmetry) of the original *soap bubble* are destroyed for most of the angles we choose. Accordingly we can speculate that if we start with a torus shaped *soap bubble*, then most of the associated surfaces should look like surfaces wrapping wildly around the torus and the symmetry of the torus should be destroyed. A torus that has constant *mean curvature* and hence is a *soap bubble* does indeed exist: it is known as the Wente torus which was discovered by H. Wente [6] and was explicitly drawn on the computer screen by various mathematicians, among them Abresch and Walter (see [1],[5]). This result would be an interesting example, since it is related to a famous open problem: whether there is a *soap film* (i.e. a surface with zero *mean curvature*) sitting completely in a bounded region in the three dimensional Euclidean space.

References

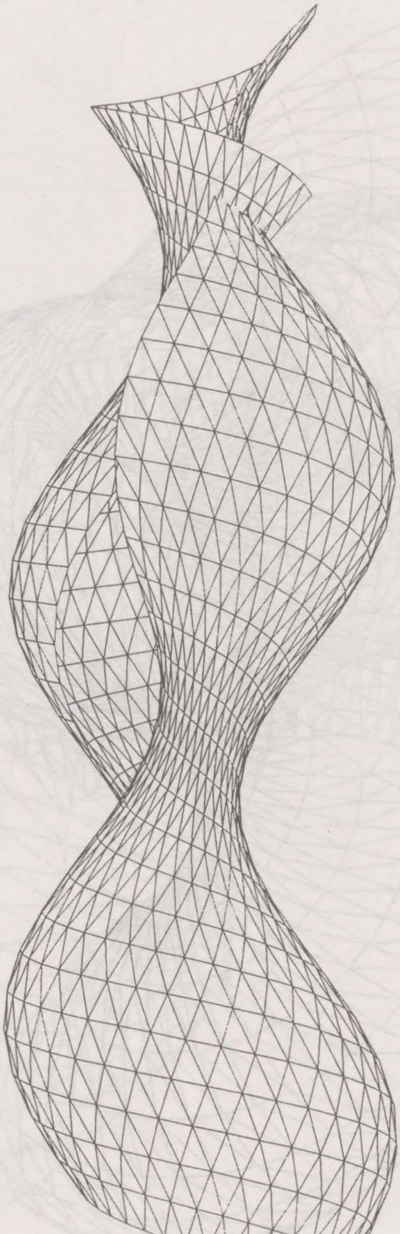
1. V. Abresch, Constant mean curvature tori in terms of elliptic functions, *J. Reine Angew. Math.*, 374 (1987), 169-172.
2. M. P. do Carmo and M. Dajezer, Helicoidal surfaces with constant mean curvature, *Tohoku M. J.*, 34 (1982), 425-435.
3. L. R. Hitt and I. M. Roussos, Computer graphics of helicoidal surfaces with constant mean curvature, (University of South Alabama, preprint).
4. H. B. Lawson, Jr., Complete minimal surfaces in S^3, *Ann. of Math.*, 92 (1970), 335-374.
5. R. Walter, Explicit examples to the H-problem of Heinz Hopf, *Geom. Dedic.*, 23 (1987), 187-213.
6. H. C. Wente, Counterexample to a conjecture of H. Hopf, *Pacific J. of Math.*, Vol. 121, No. 1 (1986), 193-243.



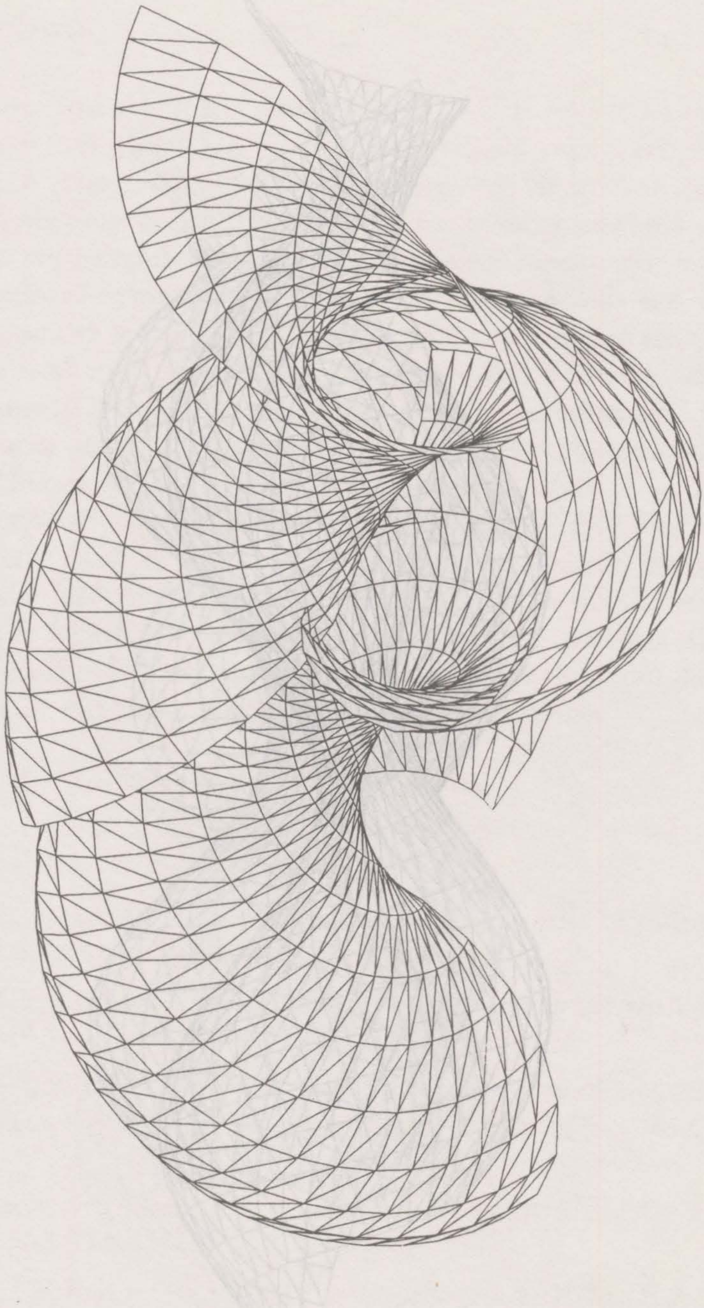
θ = 0



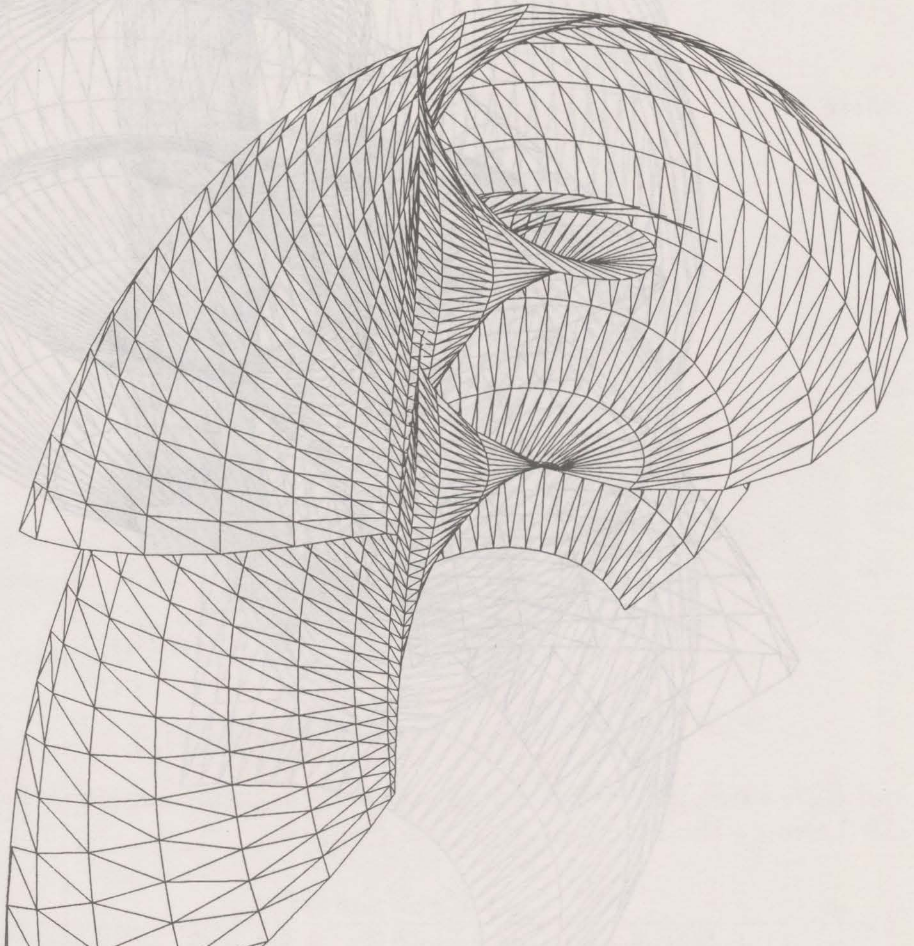
$\theta = \pi/4$



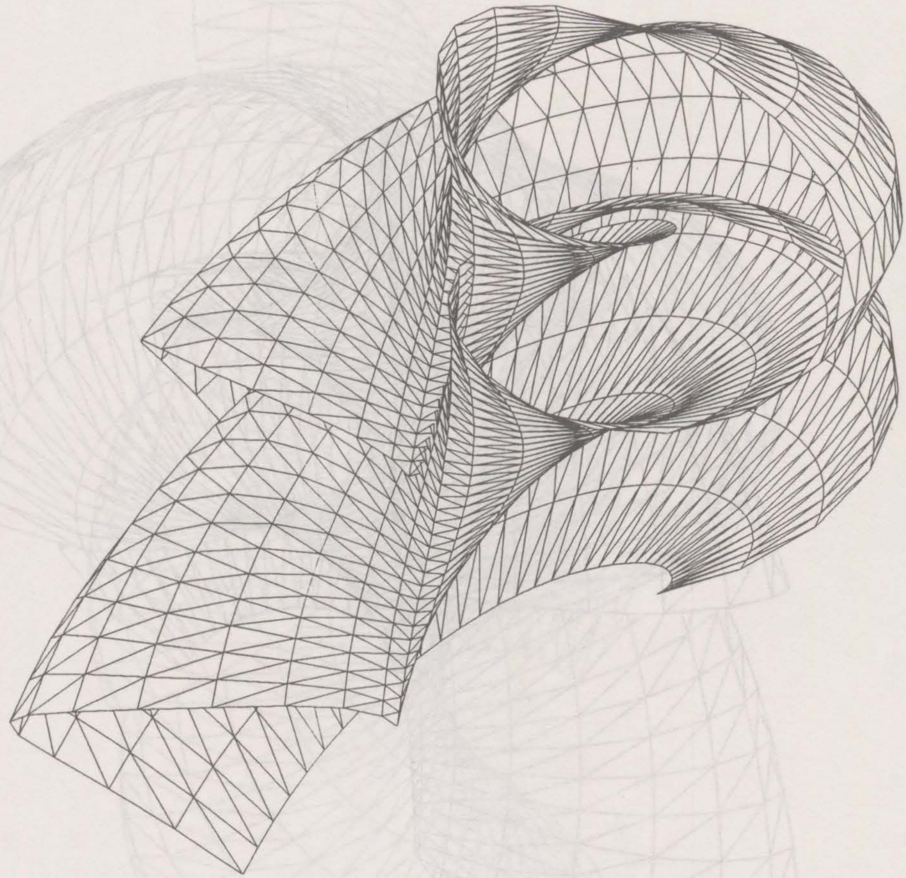
$\theta = \pi/3$



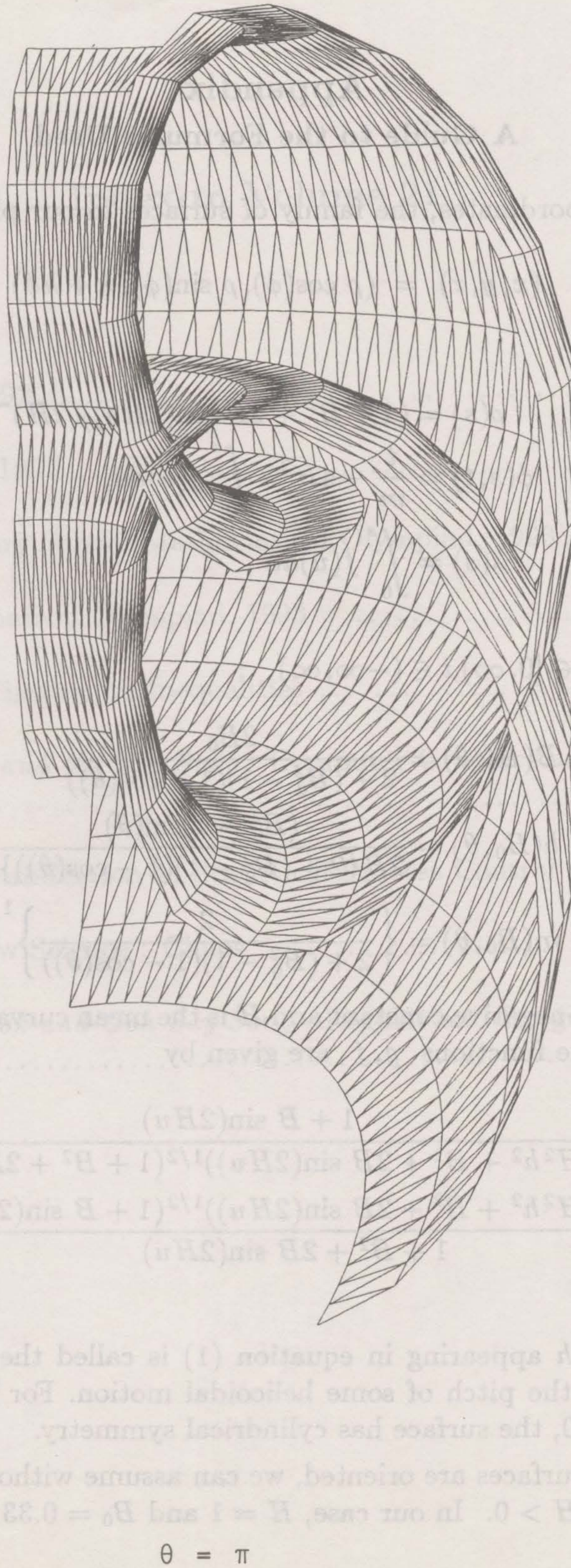
θ = π /2



$\theta = 3\pi/4$



$\theta = 4\pi/5$



θ = π

Appendix

A Guide to the Formulas Used

In Cartesian coordinates, the family of surfaces in our pictures are given by (see [3]):

$$(x, y, z) = (\rho \cos(\phi), \rho \sin(\phi), \lambda + h\phi) \tag{1}$$

where

$$\rho(s) = (1 + B^2 + 2B \sin(2Hs))(2H)^{-1} ,$$
$$\phi(s,t) = \frac{t}{m} - 4H^2 h \int_0^s g(u)du,$$
$$\lambda(s) = \int_0^s f(u)du ,$$

with

$$s \in [0, \infty), t \in (-\infty, \infty) \quad ;$$

$$B = B(B_0, \theta) = \frac{2B_0}{2 + (B_0^2 - 1)(1 - \cos(\theta))} ,$$
$$h = h(B_0, \theta) = \frac{(B_0^2 - 1)\sin(\theta)}{2H\{2 + (B_0^2 - 1)(1 - \cos(\theta))\}} ,$$
$$m = m(B_0, \theta) = \left\{ \frac{2}{2 + (B_0^2 - 1)(1 - \cos(\theta))} \right\}^{1/2} ,$$

and B_0 is a non-negative constant and H is the mean curvature, $\theta \in [0, 2\pi]$. Furthermore the functions g, f are given by

$$g(u) = \frac{1 + B \sin(2Hu)}{(1 + 4H^2h^2 + B^2 + 2B \sin(2Hu))^{1/2}(1 + B^2 + 2B \sin(2Hu))} ,$$
$$f(u) = \frac{(1 + 4H^2h^2 + B^2 + 2B \sin(2Hu))^{1/2}(1 + B \sin(2Hu))}{1 + B^2 + 2B \sin(2Hu)} .$$

Remarks.

(i) The term h appearing in equation (1) is called the pitch. It corresponds to the pitch of some helicoidal motion. For the special case when $h = 0$, the surface has cylindrical symmetry.

(ii) Since the surfaces are oriented, we can assume without loss of generality that $H > 0$. In our case, $H = 1$ and $B_0 = 0.33$.