Singapore Mathematical Society

Singapore Secondary School Mathematical Olympiad 1993

Part A

Saturday, 29 May 1993 0900–1100

Attempt as many questions as you can.
No calculators are allowed.
Enter your answers on the answer sheet provided.
No steps are needed to justify your answers.
Each question carries 3 marks.

1. Four points in space are given, not all in the same plane. How many planes can be drawn that are equidistant from these points?

2. Four girls bought a netball for \$60. The first girl paid half of the sum of the amounts paid by the other girls; the second girl paid one-third of the sum of the amounts paid by the other girls; and the third girl paid one-fourth of the sum of the amounts paid by the other girls. How much did the fourth girl pay?

3. In a series of three races, a student earns 5 points each time he wins, 3 points if he is in second place, and 1 point for third place; no ties allowed. What is the minimum number of points a student must earn in the three races to be guaranteed of earning more points than any other student?

4. An auditorium with 20 rows of seats has 10 seats in the first row and 1 more seat in each successive row. In an examination, if students are allowed to sit in any row but not next to another student in that row, what is the maximum number of students that can be seated?

5. If $x^2 - x - 1$ divides $ax^6 + bx^5 + 1$, find the sum $a + b$.

6. Let $[x]$ denote the greatest integer n such that $n \leq x$. Let $f(x) = \left[\frac{x}{12\frac{1}{2}}\right] \cdot \left[-\frac{12\frac{1}{2}}{x}\right]$. If $0 < x < 90$, then the range of f consists of k elements. Find k.

7. A piece of paper in the shape of an equilateral triangle ABC has $AB = 15$. When A is folded over to the point D on $\overline{BC}$ for which $BD = 3$, a crease is formed along a line that joins a point on $\overline{AB}$ to a point on $\overline{AC}$. The length of the crease is $\frac{1}{2}\sqrt{n}$. Find n.

8. In parallelogram $ABCD$, angle A is acute and $AB = 5$. Point E is on $\overline{AD}$ with $AE = 4$ and $BE = 3$. A line through B, perpendicular to $\overline{CD}$, intersects $\overline{CD}$ at F. If $BF = 5$, find EF.

9. A sphere is inscribed in a regular tetrahedron. If the length of an altitude of the tetrahedron is 36, find the radius of the sphere.

10. Find all pairs of positive integers m, k with $k \geq 6$, such that

$$(m+20+1)+(m+20+2)+(m+20+3)+\ldots+(m+20+k) = 1990.$$

11. There are three boxes each containing 4 white balls and 5 black balls, and all the balls are identical except in colour. A ball is transferred from the first box to the second box, then a ball is transferred from the second box to the third box, and finally a ball is transferred from the third box to the first box. Find the probability that each box will contain 4 white balls and 5 black balls again.

12. Find all three-digit numbers abc so that $a > b > c$ and $a^2 - b^2 - c^2 = a - b - c$.

13. Determine the least value of the function

$$f(x) = (x + a + b)(x + a - b)(x - a + b)(x - a - b),$$

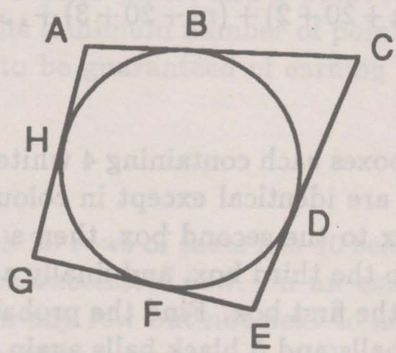
where a and b are real constants. Express your answer in terms of a and b.

14. Find all pairs of positive integers m, n such that

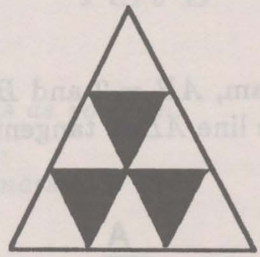
$$1! + 2! + 3! + \ldots + n! = m^2.$$

15. Each of the three three-digit numbers 153, 370, 407 is equal to the sum of the cubes of its digits. Find another such three-digit number.

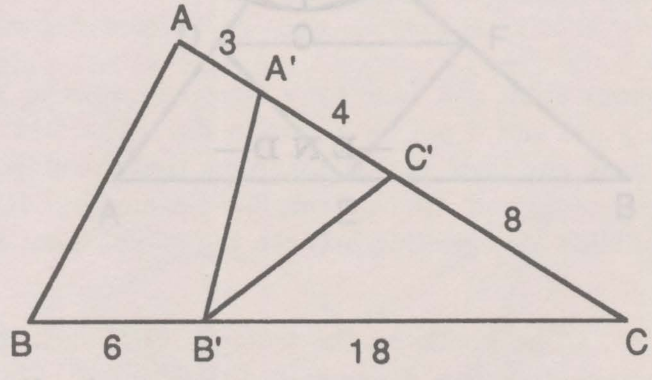
16. $ACEG$ is a quadrilateral circumscribing a circle and tangent to the circle at the points B, D, F, and H, with B lying on $\overline{AC}$, D lying on $\overline{CE}$, F lying on $\overline{EG}$, and H lying on $\overline{GA}$. If $AB = 3$, $CD = 4$, $EF = 5$, and $GH = 6$, find the radius of the circle.



17. An equilateral triangle ABC of side 1 is subdivided into 9 equilateral triangles, and 3 of the smaller triangles are shaded as shown. Repeat the process indefinitely for the unshaded triangles. Find the area of the shaded region.



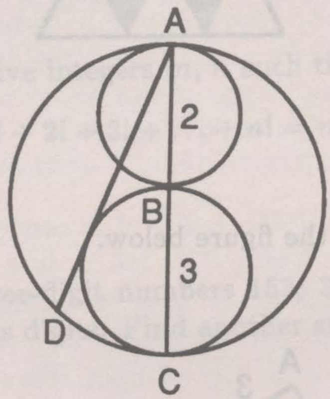
18. Find $\dfrac{\Delta A'B'C'}{\Delta ABC}$ for the figure below.



19. If $a > b$ are positive integers, solve

$$\frac{a+b}{a-b} > ab.$$

20. In the following diagram, $AB = 2$ and $BC = 3$ are the diameters of the circles shown. The line AD is tangent to the circle of diameter 3. Find BD.



—E N D—

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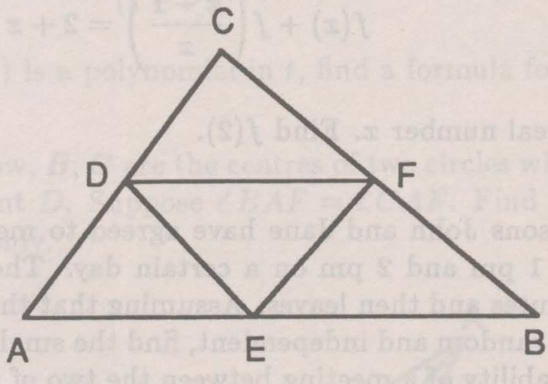
Part B

Saturday, 29 May 1993 1100–1200

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 20 marks.

1. In the triangle ABC, $DF = EF = ED$ and $DC = FB = EA$. Prove that triangle ABC is equilateral.



2. Show that if a, b, c are three distinct positive integers, then there will be at least two of them, say a and b, such that $a^3 b - ab^3$ is a multiple of 30.

—E N D—