Singapore Mathematical Society

Singapore Mathematical Olympiad 1993

Part A

Saturday, 19 June 1993 0900–1000

Attempt as many questions as you can.
No calculators are allowed.
Enter your answers on the answer sheet provided.
No steps are needed to justify your answers.
Each question carries 5 marks.

1. Let $f(x)$ be a function such that

$$f(x) + f\left(\frac{x-1}{x}\right) = 2 + x$$

for any real number x. Find $f(2)$.

2. Two persons John and Jane have agreed to meet at a definite spot between 1 pm and 2 pm on a certain day. The first to come waits for t minutes and then leaves. Assuming that the arrival of John and Jane are random and independent, find the smallest value of t so that the probability of a meeting between the two of them is at least 50%.

3. Let A', B', C' be points on the triangle ABC such that $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are angle bisectors. Suppose $\angle A'C'B' = 90^\circ$. Find $\angle ACB$.

4. How many distinct numbers are there in the sequence

$$\left[\frac{1^2}{1993}\right], \left[\frac{2^2}{1993}\right], \ldots, \left[\frac{1993^2}{1993}\right],$$

where $[n]$ denotes the largest integer $\leq n$.

5. A lady made three circular doilies with radii of 2, 3, and 10 inches, respectively. She placed them on a circular table so that each doily touched the other two. If each doily also touched the edge of the table, what was the radius of the table?

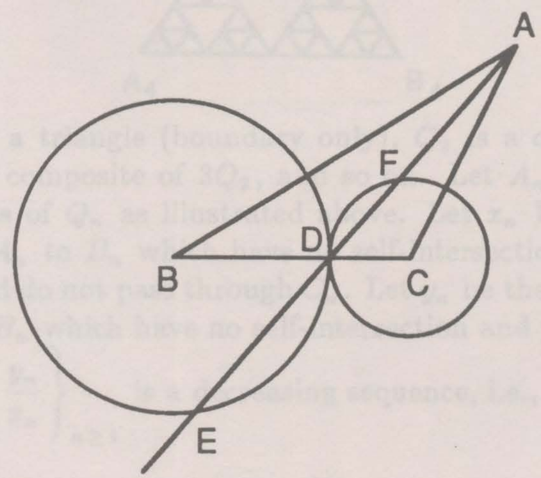
6. There are 600 people, numbered consecutively from 1 to 600, standing in a circle. First, #2 sits down, then #4, #6, etc., and this continues around the circle, with every other standing person sitting down until just one person is left standing. What is the number of the person standing? (For example, if there are 6 persons, the seating order is 2, 4, 6, 3, 1, and 5 is left standing).

7. If we write the n$^{\text{th}}$ derivative (with respect to t) of $f(t) = \dfrac{1}{1-t^2}$ as

$$f^{(n)}(t) = \frac{H_n(t)}{(1-t^2)^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

where each $H_n(t)$ is a polynomial in t, find a formula for $H_n(1)$.

8. In the figure below, B, C are the centres of two circles which intersect at the single point D. Suppose $\angle BAF = \angle CAF$. Find AD in terms of AE and AF only.



9. Let S be the set consisting of 8-place decimals between 0 and 1 of the form $0.x_1x_2x_3x_4x_5x_6x_7x_8$ where $1 \leq x_i \leq 6$ for $1 \leq i \leq 8$. Find a number α in S such that there is a 50% chance that a randomly selected number β from S is less than or equal to α.

10. Let ABC be a right-angled triangle with $\angle C = 90°$. Let the bisectors of $\angle A$ and $\angle B$ intersect $\overline{BC}$ and $\overline{CA}$ at D and E respectively. Given that $CD = 9$ and $CE = 8$, find the lengths of the sides of ABC.

—E N D—

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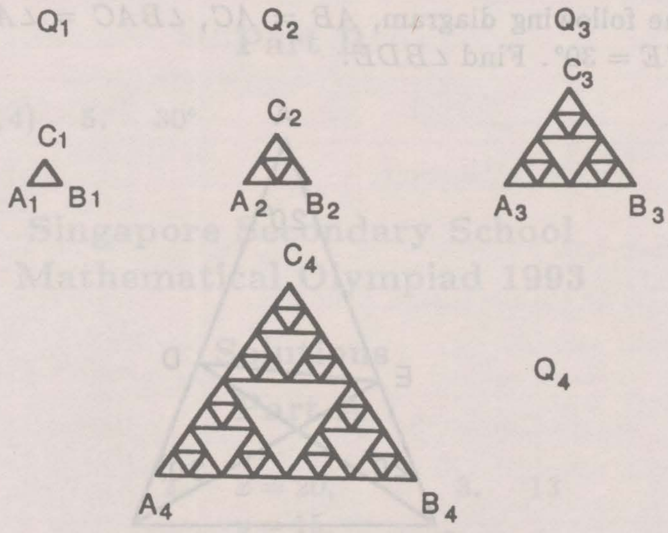
Part B

Saturday, 19 June 1993 1000–1200

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 25 marks.

1. Consider the figures $Q_1, Q_2, \ldots$ below:



where Q_1 is a triangle (boundary only), Q_2 is a composite of $3Q_1$ and Q_3 is a composite of $3Q_2$, and so on. Let A_n, B_n, C_n be the three vertices of Q_n as illustrated above. Let x_n be the number of paths from A_n to B_n which have no self-intersection (i.e., there are no loops) and do not pass through C_n. Let y_n be the number of paths from A_n to B_n which have no self-intersection and pass through C_n.

Show that $\left\{ \dfrac{y_n}{x_n} \right\}_{n \geq 1}$ is a decreasing sequence, i.e.,

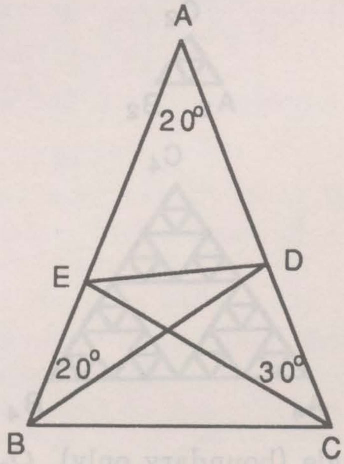
$$\frac{y_1}{x_1} \geq \frac{y_2}{x_2} \geq \frac{y_3}{x_3} \geq \cdots .$$

2. Find all pairs of positive integers (x, y) such that $x^y = y^{x-2y}$. Justify your answer.

3. Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree n, where $a_0, a_1, \ldots, a_{n-1}$ are real numbers. Show that there exists an integer j between 1 and $n+1$, such that $f(j)$ has absolute value not less than $\frac{n!}{2^n}$.

4. Show that if γ and t are rational numbers satisfying $\gamma = \cos t\pi$, then $\gamma = 0, \pm\frac{1}{2}, \pm 1$.

5. In the following diagram, $AB = AC$, $\angle BAC = \angle ABD = 20°$ and $\angle ACE = 30°$. Find $\angle BDE$.



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Singapore Mathematical Olympiad 1993

Solutions

Part A

1.	$5/4$	**2.**	17.6	**3.**	$2\pi/3$
4.	1495	**5.**	$15in$	**6.**	177
7.	$2^n n!$	**8.**	$\sqrt{AE \cdot AF}$	**9.**	0.36666666
10.	$AB = 30$, $AC = 18$, $BC = 24$				

Part B

2.	$(1,1),(16,4)$	**5.**	30°

Singapore Secondary School Mathematical Olympiad 1993

Solutions

Part A

1.	7	**2.**	$x = 20$, $y = 15$, $z = 12$	**3.**	13
4.	200	**5.**	3	**6.**	$0, -1, -2, \ldots, -7$
7.	$n = 343$	**8.**	$\sqrt{10}$	**9.**	9
10.	$k = 20$, $m = 69$	**11.**	$14/45$	**12.**	764 or 976
13.	$-4a^2b^2$	**14.**	$n = m = 1$, $n = m = 3$	**15.**	371
16.	$\sqrt{19}$	**17.**	$\sqrt{3}/4$	**18.**	$1/5$
19.	$a = 2$, $b = 1$	**20.**	$6/7$		

Singapore Mathematical Society

Singapore Secondary School

Mathematical Olympiad 1993

Announcement of Results

A total of 507 students from 50 secondary schools took part in the Singapore Secondary School Mathematical Olympiad on Saturday, 29 May 1993, at Victoria Junior College. The team results and individual results are as follows.

Team Results

Position	School	Team Members
1	Raffles Institution	Chor Han Ping, Davin Chua Choong Tze Teoh Yit Pang, Kevin
2	The Chinese High School	Ng Say Kai Teo Yi-Wei Yap Choon Hwai
3	Anglo-Chinese School	Herman Chow Huang Hede Xu Weifa
4	Dunman High School	Chin Jit Kee Lim Hiu Fai Soo Kong Hua
5	Victoria School	Lee Wee Beng Tan Chia Seng, Albert Yudi Praroto

A list of the top fifteen schools who have participated in the competition is as follows.

1. Raffles Institution
2. The Chinese High School
3. Anglo-Chinese School
4. Dunman High School

5. Victoria School
6. Raffles Girls' Secondary School
7. River Valley High School
8. St. Joseph's Institution
9. Bendemeer Secondary School
10. Gan Eng Seng School
11. Paya Lebar Methodist Girls' Secondary School
12. Nan Hua Secondary School
13. Yishun Town Secondary School
14. CHIJ Secondary (Toa Payoh)
15. Singapore Chinese Girls' School

The winning school, Raffles Institution, will be awarded the Singapore Mathematical Society Challenge Trophy and certificates will be given to the top 15 schools. Prizes will also be given to members of the first five ranking schools listed above.

Individual Results

Position	Competitor	School
1	Chor Han Ping, Davin	Raffles Institution
2	Xu Weifa	Anglo-Chinese School
3	Ng Say Kai	The Chinese High School
4	Yap Choon Hwai	The Chinese High School
5	Chua Choong Tze	Raffles Institution
6	Teo Yi-Wei	The Chinese High School
7	Cheng Hoi Wai	The Chinese High School
8	Fong Kah Weng	The Chinese High School
9	Herman Chow	Anglo-Chinese School
10	Teoh Yit Pang, Kevin	Raffles Institution

The following receive honourable mentions: Kang Chun Siong (The Chinese High School), Huang Hede (Anglo-Chinese School), Mervyn Tan (Anglo-Chinese School), Jeremy J. Samuel (The Chinese High School), Au Yeung Chun Yiu, Adrian (Raffles Institution), Luo Chengwei, Adrian (Raffles Institution), Chin Jit Kee (Dunman High School), Kho Yew Hann (The Chinese High School), Thevendran Senkodan (Raffles Institution), Yudi Praroto (Victoria School).

The individual winner, Chor Han Ping of Raffles Institution, will be awarded the Liew Mai Heng Memorial Prize. Prizes will also be awarded to the next nine ranking individuals. Certificates will be awarded to those who receive honourable mentions.

Singapore Mathematical Society

Singapore Mathematical Olympiad 1993

Announcement of Results

A total of 116 students from 10 junior colleges and 9 secondary schools took part in the Singapore Mathematical Olympiad on Saturday, 19 June 1993, at the National University of Singapore. The results are as follows.

Team Results

Position	School	Team Members
1	Hwa Chong Junior College	Gan Wee Liang Lim Chu Wee Ng Chee We
2	Raffles Junior College	Ho Chui Ping, Tracey Huang Jingxiang Tan Choon Siang
3	Raffles Institution	Au Yeung Chun Yiu, Adrian Chor Han Ping, Davin Thevendran Senkodan

The winning school, Hwa Chong Junior College, will be awarded the Singapore Mathematical Society Challenge Shield and the team members will be awarded the Southeast Asian Mathematical Society Prize. Prizes will also be given to members of the schools ranked second and third.

Individual Results

Position	Competitor	School
1	Lim Chu Wee	Hwa Chong Junior College
2	Gan Wee Liang	Hwa Chong Junior College
3	Tan Choon Siang	Raffles Junior College
4	Ng Chee We	Hwa Chong Junior College
5	Chor Han Ping, Davin	Raffles Institution

The following receive honourable mentions: Chong Hooi Min (Hwa Chong Junior College), Goh Kim Hua, Alan (Victoria Junior College), Chin Siong Ngiap (Hwa Chong Junior College), Lo Tong Jen (Victoria Junior College), Thevendran Senkodan (Raffles Institution).

The individual winner, Lim Chu Wee of Hwa Chong Junior College, will be awarded the Singapore Mathematical Society Prize. Prizes will also be awarded to the next four ranking individuals. Certificates will be awarded to those who receive honourable mentions.