Problems and Solutions

Problems and Solutions. The aim of this section is to encourage readers to participate in the intriguing process of problem solving in Mathematics. This section publishes problems and solutions proposed by readers and editors.

Readers are welcome to submit solutions to the following problems. Your solutions, if chosen, will be published in the next issue, bearing your full name and address. A publishable solution must be correct and complete, and presented in a well-organised manner. Moreover, elegant, clear and concise solutions are preferred.

Readers are also invited to propose problems for future issues. Problems should be submitted with solutions, if any. Relevant references should be stated. Indicate with an ° if the problem is original and with an * if its solution is not available.

All problems and solutions should be typewritten double-spaced, and two copies should be sent to the Editor, Mathematical Medley, c/o Department of Mathematics, The National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

Problems

The following problems are questions in the International Mathematical Olympiad held in Moscow, 1992.

P20.2.1. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

P20.2.2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x^2 + f(y)\big) = y + \big(f(x)\big)^2 \quad \text{for all } x, y \text{ in } \mathbb{R}.$$

P20.2.3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

P20.2.4. In the plane let C be a circle, L a line tangent to he circle C and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

P20.2.5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

where $|A|$ denotes the number of elements in the finite set A. (Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane).

P20.2.6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

Solutions

P20.1.1. *Solution by the Editor*

We may assume without loss of generality that f is an even functions because in general we can write

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \frac{f(x) + f(-x)}{2} \quad \text{is even,}$$

$$f_2(x) = \frac{f(x) - f(-x)}{2} \quad \text{is odd,}$$

$f_1(\pm 1) = 0$, $f_1^{(4)}(x)$ is continuous, $|f_1(x)| \leq 1$, for all $x \in [-\sqrt{3}, \sqrt{3}]$ and

$$\int_{-\sqrt{3}}^{\sqrt{3}} f(x)dx = \int_{-\sqrt{3}}^{\sqrt{3}} f_1(x)dx.$$

By Taylor's formula with integral remainder,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{1}{3!}\int_0^x (x-t)^3 f^{(4)}(t)dt.$$

The remainder term can be written as

$$\frac{1}{4!}\int_0^{\sqrt{3}} (x-t)_+^3 f^{(4)}(t)dt,$$

where

$$(x-t)_+^3 = \begin{cases} (x-t)^3 & \text{if } x \geq t \\ 0 & \text{if } x < t. \end{cases}$$

Since f is even, $f'(0) = f^{(3)}(0) = 0$. Hence

$$f(x) = f(0) + \frac{f''(0)}{2!}x^2 + \frac{1}{3!}\int_0^{\sqrt{3}} (x-t)_+^3 f^{(4)}(t)dt. \tag{1}$$

In particular, with $x = 1$,

$$0 = f(1) = f(0) + \frac{f''(0)}{2!} + \frac{1}{3!}\int_0^{\sqrt{3}} (1-t)_+^3 f^{(4)}(t)dt. \tag{2}$$

Subtracting (2) from (1) gives

$$f(x) = \frac{f''(0)}{2!}(x^2-1) + \frac{1}{3!}\int_0^{\sqrt{3}} \left\{ (x-t)_+^3 - (1-t)_+^3 \right\} f^{(4)}(t)dt.$$

Integrating the above equation leads to

$$\int_0^{\sqrt{3}} f(x)dx = \frac{1}{3!}\int_0^{\sqrt{3}}\int_0^{\sqrt{3}} \left\{ (x-t)_+^3 - (1-t)_+^3 \right\} f^{(4)}\,dtdx \tag{3}$$

since

$$\int_{-\sqrt{3}}^{\sqrt{3}} (x^2-1)dx = 0.$$

Equation (3) can be expressed as

$$\int_0^{\sqrt{3}} f(x) = \frac{1}{3!}\int_0^{\sqrt{3}}\int_0^{\sqrt{3}}\left\{(x-t)_+^3 - (1-t)_+^3\, dx\right\} f^{(4)}(t)dt$$

$$= \frac{1}{3!}\int_0^{\sqrt{3}}\left\{\frac{(\sqrt{3}-t)_+^4}{4} - \sqrt{3}(1-t)_+^3\right\} f^{(4)}(t)dt.$$

Hence

$$\left|\int_0^{\sqrt{3}} f(x)\right| \le \frac{1}{3!}\int_0^{\sqrt{3}}\left|\frac{(\sqrt{3}-t)_+^4}{4} - \sqrt{3}(1-t)_+^3\right| \left|f^{(4)}(t)\right| dt$$

$$\le \frac{1}{3!}\int_0^{\sqrt{3}}\left|\frac{(\sqrt{3}-t)_+^4}{4} - \sqrt{3}(1-t)_+^3\right| dt,$$

since $|f^{(4)}(t)| \le 1$ for all $t \in [-\sqrt{3}, \sqrt{3}]$. It is easy to check that

$$\frac{1}{4}(\sqrt{3}-t)_+^4 - \sqrt{3}(1-t)_+^3 \ge 0.$$

Hence

$$\left|\int_0^{\sqrt{3}} f(x)dx\right| \le \frac{1}{3!}\int_0^{\sqrt{3}}\left\{\frac{(\sqrt{3}-t)_+^4}{4} - \sqrt{3}(1-t)_+^3\right\} dt$$

$$= \frac{1}{3!}\left\{\frac{(\sqrt{3})^5}{5\times 4} - \frac{\sqrt{3}}{4}\right\}$$

$$= \frac{\sqrt{3}}{3!5}.$$

Finally

$$\left|\int_{-\sqrt{3}}^{\sqrt{3}} f(x)dx\right| = 2\left|\int_0^{\sqrt{3}} f(x)dx\right| \le \frac{\sqrt{3}}{15}.$$

The bound $\frac{\sqrt{3}}{15}$ is best possible, because the function $f(x) = \frac{1}{4!}(x^4 - 1)$ satisfies all the conditions of the problem and

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{4!}(x^4 - 1)dx = \frac{\sqrt{3}}{15}.$$