Problem-Solving Strategies: Research Findings from Mathematics Olympiads

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Introduction

The starting point for work in problem-solving strategies in mathematics is, as we all know, George Pòlya. His famous books laid the foundation for research in heuristics - strategies and techniques for making progress on unfamiliar and nonstandard problems. He was also the first person to describe problem-solving strategies in such a way that they could be taught. In his pioneer book *How To Solve It*, published in 1945, Pòlya proposed four phases in problem-solving: *understanding the problem, devising a plan, carrying out the plan* and *looking back.*

In his four-phase plan and other works, Pòlya suggested a list of heuristics, which includes:

Drawing a figure
Introducing suitable notations, auxiliary elements
Examining special cases (looking at simpler cases to search for a pattern; examining limiting cases to explore the range of possibilities)
Modifying the problem (replacing given conditions by equivalent ones; recombining the elements of the problem in different ways)
Exploiting related problems (simpler problems, auxiliary problems, analogous problems)
Working backwards
Arguing by contradiction or contrapositive
Decomposing and recombining
Generalising
Specialising

Exploiting symmetry and parity

In the 1980s, Alan Schoenfeld proposed a framework for investigation of complex mathematical problem-solving behaviour. The framework comprises four categories: **resouces** (mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand), **heuristics, control** (global decisions regarding the selection and implementation of resources and strategies) and **belief systems** (one's perspectives regarding the nature of mathematics and how one goes about working it). This paper reports the author's pilot study on problem-solving strategies (mainly heuristics but some useful resources and guidelines on control in the context of Schoenfeld's framework are also included) for problems in national and international olympiads. This report is a brief summary of my findings after investigating the solutions to a pool of olympiad problems with one example included to illustrate each strategy. The survey is not exhaustive but it is a first step towards the author's intended large-scale study on the topic. In completing the report, I was benefitted by the works of a number of contemporary Chinese problemists, to whom I wish to express my gratitude.

Some Basic Strategies

One of the most commonly used strategies in solving olympiad problems is to **search for a pattern**, from which we may be able to make a conjecture and then prove it.

[IMO 1964]
Find all positive integers n for which $2^n - 1$ is divisible by 7.

If we calculate the values of 2^n for $n = 1, 2, 3, ...,$ up to 10 (say) and divide each value by 7, then the pattern of remainders obtained is clear enough to help us to form a conjecture on the remainders and then prove it by using simple properties of integers and, perhaps, the binomial theorem. Some easier olympiad problems such as this one can be solved with little further difficulty after we have observed a pattern. However, there are even more olympiad problems for which the heuristic 'search for a pattern' provides only the first breakthrough and proving the conjecture is still challenging enough.

Another useful heuristic in solving olympiad problems is to **modify the problem**. This heuristic is so general that it may not be of great help unless we know how to modify the given problem after noting the characteristics of the problem. Survey of expert solutions to olympiad problems indicates several possible strategies within this heuristic. The first and the most common one is to **solve a simpler but similar problem.** When the numbers in the problems are unnecessarily large, such as using the current year as a crucial number, then we may replace them by small numbers with little disturbance to the structure of the problem. In most of the cases, such a replacement will lead us to search for a pattern which can be generalised to provide a solution for the original problem with larger numbers.

[IMO 1989]
Prove that the set $\{1, 2, ..., 1989\}$ *can be expressed as the disjoint union of subsets* $A_i\,(i = 1, 2, ..., 117)$ *such that*
(i) *each* A_i *contains* 17 *elements, and*
(ii) *the sum of all the elements in each* A_i *is the same.*

The number 1989 in this problem is not significant at all. We may replace it by any composite odd number which is the product of two moderately (compared to the product) large factors. We may replace 1989 by 35, 117 by 5, and 17 by 7. Then we try to partition the simplified set $\{1, 2, ..., 35\}$. After some trials, we should be able to observe that the problem will be much easier if the number 1989 is replaced by an *even* number (such as 28) instead of an *odd* number like 35. After **solving the more accessible related problem** by replacing 1989 by 28, 117 by 4, and 17 by 7, we should try to build up our solution to the problem on $\{1, 2, ..., 35\}$ upon some modification to the solution of the previous one while keeping as much of the pattern as possible. If we are through with 35, then the case of 1989 is straightforward by **analogy**. The rest are just technical jargons to make the argument rigorous.

The second common strategy of modifying the given problem is to **restate the problem in an equivalent form** which is easier to handle.

[Kürschàk 1968-69]
Prove that if every element, starting from the second one, of an infinite sequence of natural numbers is equal to the harmonic mean of its neighbours, then all the elements of the sequence are equal.

After **writing down the given condition using mathematical symbols**, we should be aware that the problem can be restated in an alternative way which is more familiar to us:

Prove that if $\{a_n\}$ is a sequence of natural numbers such that $\{1/a_n\}$ is an arithmetic sequence, then all the a_n 's are equal.

This simplified version should then enable us to solve the problem easily using simple properties of natural numbers.

Some olympiad problems may be modified to enable them to be solved over wider domains or under stronger conditions than those given ones.

[IMO 1975]

Let $a_1, a_2, a_3, \ldots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x, y positive integers and $q > p$.

Here the four natural numbers are all undetermined, thus making the problem difficult because of the many degrees of freedom. We may, as a trial, fix the value of one of these four parameters. If we assign $x = 1$, then the problem is modified to a stronger one which is easier to handle:

Prove that there are infinitely many a_m satisfying

$$a_m \equiv a_p \pmod{a_q}$$

with $p \neq q$.

This modified problem can be solved by using congruence classes and the given contraints.

Symmetry in some olympiad problems also help us to solve the problems. Some problems, as in the following one, can be simplified by proving only one of the several symmetric conditions.

[USA 1974]

Prove that if a, b and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

The inequality required, after rearrangement, is equivalent to

$$\left(\frac{a}{b}\right)^{\frac{a-b}{3}} \left(\frac{b}{c}\right)^{\frac{b-c}{3}} \left(\frac{c}{a}\right)^{\frac{c-a}{3}} \geq 1.$$

By symmetry, it suffices to prove that

$$\left(\frac{a}{b}\right)^{\frac{a-b}{3}} \geq 1.$$

We may also use symmetry to fix indeterminate factors in the problem.

[Kürschák 1942-43]
If $a < b < c < d$ and (x, y, z, t) is a permutation of (a, b, c, d), how many different values can be attained by the expression

$$n = (x-y)^2 + (y-z)^2 + (z-t)^2 + (t-x)^2.$$

Because of symmetry in the expression $n + (x-z)^2 + (y-t)^2$, we need only consider the expression $(x-z)^2 + (y-t)^2$, which equals

$$x^2 + y^2 + z^2 + t^2 - 2(xz + yt).$$

The first part of this expression is also symmetric, so we need only consider the simple expression $xz + yt$.

Methods of Proof

There are several general methods of proof useful for olympiad problems. More sophisticated approaches will be left to later sections. Since there are so many olympiad problems which are solved by the **proof by contradiction**, there is no need for me to give any example here. However, I would like to generalise observations on olympiad problems which may be solved by using the proof by contradiction. A lot of these problems fall into one of the following two categories:

1. The consequent in the proposition appears in the form of the negation of a statement.
2. Keywords such as 'at most', 'at least', unique', 'concurrent', 'collinear', 'coplanar' appear in the consequent of the proposition.

The next common method of proof for olympiad problems is the **proof by exhaustion**. Very often we have no better choice than listing all possible cases and proving each one of them or disproving each of the possible cases constituting the negation of the conclusion.

[China 1983]
Among all tetrahedrons of lengths of sides 2, 3, 3, 4, 5, 5, which has the largest volume? Prove your assertion.

The criterion for forming a tetrahedron is that the sides must form all the triangular faces of the tetrahedron. Since the difference of any two sides of a triangle must be less than the third side, let us focus on those triangular faces with a side of length 2. For these triangles, there are only four possible combinations for the two remaining sides. Furthermore, there can only be three possible cases for the two adjacent triangles with a common side of length 2. Find the volume for each of these three cases and compare with one another.

Proof by induction is popular among olympiad problems and we can easily find a large number of examples to illustrate both simple induction and strong induction. However, for some olympiad problems, we must apply a more sophisticated version of induction - **spiral induction:**

Let $\{P_n\}$ and $\{Q_n\}$ be two sequences of propositions on natural numbers n. If

(i) P_1 is true, and

(ii) for any natural number k, P_k is true $\Rightarrow$ Q_k is true $\Rightarrow$ P_{k+1} is true,

then P_n and Q_n are true for every natural number n.

Spiral induction can be regarded as a combination of simple induction and strong induction. It can be extended easily to more than two sequences of propositions.

[Putnam 1956]
Consider a set of $2n$ points in space, $n > 1$. Suppose they are joined by at least n^2+1 segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangles being formed.

Let us denote the proposition in the first part of the problem by P_n and construct its partner Q_n: if there are $2n-1$ $(n \geq 2)$ points in space and these points are connected by $n(n-1)+1$ segments, then these segments form at least one triangle. Q_1 is obviously true. The implications $Q_k \Rightarrow P_k \Rightarrow Q_{k+1}$ can be established naturally by removing the point joined by the least number ($\leq k$) of segments in each case. The second part of the problem can be proved by dividing the $2n$ points into two sets of n points each such that any two points in different sets are joined by a segment while no points in the same set are joined.

In proving propositions, the importance of elements involved is often unbalanced. Some extremal elements possess properties not present in others. These properties may facilitate the solution of the problem. The following example demonstrates the use of extremal elements in problems on existence.

[Poland 1975]

A sequence $\{a_n\}$ satisfies the following properties:

There exists a natural number n such that

$$a_1 + a_2 + \dots + a_n = 0$$

and $$a_{n+k} = a_k, \quad k = 1, 2, \dots .$$

Prove that there exists a natural number N such that, for any $k = 1, 2, \dots$,

$$\sum_{i=N}^{N+k} a_i \geq 0.$$

Let $S_j = a_1 + a_2 + \dots + a_j$ $(j = 1, 2, \dots)$.

Note that $S_{pn} = 0$ $(p = 1, 2, \dots)$ and $S_{n+j} = S_j$ $(j = 1, 2, \dots)$, which shows that there are finitely many terms in $\{S_j\}$. Denote by S_m the least element among $S_1, S_2, \dots, S_n$. Show that $N = m + 1$ satisfies the requirement of the problem.

The use of extremal elements also helps to prove some propositions by contradiction.

[China 1988]

If $a_1 = 1$, $a_2 = 2$, and

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n & \text{when } a_n a_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{when } a_n a_{n+1} \text{ is odd,} \end{cases}$$

prove that, for any natural number n, $a_n \neq 0$.

After calculating the first few values of a_n, we should have noticed the pattern about parity of the terms:

odd, even, odd, odd, even, odd, odd, even, odd, odd, ...

Let m be the *least* value that a_m is zero. Then

$$a_m = \dots = 4a_{m-2} - 3a_{m-3}.$$

This shows that a_{m-3}, and hence $a_{m-6}, a_{m-9}, \dots, a_2$, will be multiples of 4. Since a_2 is not a multiple of 4, we have a contradiction.

Strategies on Counting

Problems on counting are popular in every olympiad. Among the solutions to the numerous olympiad problems related to counting, four strategies are found to be most useful. The pigeonhole principle, which is so fundamental and widely known, is still very useful in solving olympiad problems. The following IMO problem illustrates how crucial the principle is in solving even problems not explicitly related to counting.

[IMO 1987]
Let $x_1, x_2, ..., x_n$ be real numbers satisfying

$$x_1^2 + x_2^2 + ... + x_n^2 = 1.$$

Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, ..., a_n$, not all zero, such that $|a_i| \leq k-1$ for all i and

$$|a_1 x_1 + a_2 x_2 + ... + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

The number of expressions of the form $b_1 x_1 + b_2 x_2 + ... + b_n x_n$, where $b_1, b_2, ..., b_n$ are integers such that $0 \leq b_i \leq k-1$ for all i, is k^n. Since the denominator of the R.H.S. of the inequality is $k^n - 1$, we should try to prove that

$$|b_1 x_1 + b_2 x_2 + ... + b_n x_n| \leq (k-1)\sqrt{n}.$$

The expressions k^n and $k^n - 1$ hint us to use the pigeonhole principle to complete the proof. The sum of products in the inequality and the sum of squares in the problem suggest us to use Cauchy's inequality (and simple properties of absolute values). Finally, we divide the interval $[0, (k-1)\sqrt{n}]$ into $k^n - 1$ parts of equal widths and then apply the pigeonhole principle to the k^n expressions. Set a_i to be the difference of the coefficients of x_i in the two 'pigeons' in the same hole and we are through.

The second useful tool in counting is the inclusion-exclusion principle, which states that

$$\begin{aligned} &|A_1 \cup A_2 \cup ... \cup A_n| \\ &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - ... \\ &\quad + (-1)^{n-1} |A_1 \cap A_2 \cap ... \cap A_n|. \end{aligned}$$

[Poland 1968]

Let Z be a set of $n(> 3)$ points on a plane. No three of the points are collinear. If every point in Z is connected by line segments to at least k points in Z, where $n/2 < k < n$, prove that these line segments form at least one triangle.

Let AB be one of the line segments. Denote by S and T respectively the sets {points in Z which are connected to $A\} \setminus \{B\}$ and {points in Z which are connected to $B\} \setminus \{A\}$. The proposition is equivalent to $|S \cap T| > 0$.

The third principle is not as well-known as the previous two but it is very useful in solving some olympiad problems on counting. This principle is named after **Fubini** and may be stated in different versions, one of which is as follows:

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$. If $S \subset A \times B$, then

$$|S| = \sum_{i=1}^{m} |S_{(a_i,*)}| = \sum_{j=1}^{n} |S_{(*,b_j)}|.$$

In applying the Fubini principle to solve olympiad problems, the sets A, B and S must be constructed by the solver using the given conditions in the problem. The construction of S is most crucial and is a decision on a rule of matching elements of A and B.

[Moscow 1960]

There are m points on a plane. Some of the points are connected by line segments. Each point is connected to exactly l line segments. Find the possible values of l.

Let $\{a_1, a_2, ..., a_m\}$ be the set of the m points and $\{b_1, b_2, ..., b_n\}$ be the set of the n line segments. Apply the Fubini principle to the set of all ordered pairs (a_i, b_j) such that a_i is an endpoint of b_j. Note that $l < m$.

It is often a formidable task to count the number of elements in a set in usual ways. If we can establish a **one-one correspondence** between this set and another set such that the number of elements in the other set can be counted easily, then the problem can be solved.

[Putnam 1956]

Given n objects arranged in a row. A subset of these objects is called unfriendly if no two of its elements are consecutive. Show that the *number of unfriendly subsets each having k elements is*

$$\binom{n-k+1}{k}.$$

Let the row of objects be $A(1), A(2), ..., A(n)$. Denote by $\{A(i_1), A(i_2), ..., A(i_k)\}$ $(i_1 < i_2 < ... < i_k)$ an unfriendly subset. The sequence $i_1, i_2 - 1, i_3 - 2, ..., i_k - (k-1)$ is obviously a sub-sequence of $1, 2, 3, ..., n - (k-1)$. Conversely, for every sub-sequence $j_1, j_2, ..., j_k$, the set $\{A(j_1), A(j_2 + 1), A(j_3 + 2), ..., A(j_k + (k-1))\}$ is unfriendly. Hence we have established a one-one correspondence between the set of unfriendly k-element subsets and the set of k-term sub-sequences of $1, 2, 3, ..., n - k + 1$.

In this solution, we transform the counting of elements under constraints to the counting of elements in another set, under no contraints, through one-one correspondence.

Synthesis versus Analysis

In proving inequalities, two basic approaches exist: synthesis and analysis. These two approaches are not exhaustive but serve the purpose of attacking a wide range of inequalities in olympiad problems. If we follow the synthetic approach, then we start from given conditions and well-known inequalities to deduce the required inequality using basic properties of inequalities.

[China 1989]

Let $x_1, x_2, ..., x_n$ $(n \geq 2)$ be positive numbers such that

$$\sum_{i=1}^{n} x_i = 1,$$

prove that

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \geq \frac{\sum_{i=1}^{n} \sqrt{x_i}}{\sqrt{n-1}}.$$

Our task will be easier if we can separate the x_i's in the numerator and the denominator of the summand at the L.H.S. After some trials, we can arrive at

$$\frac{x_i}{\sqrt{1-x_i}} = \frac{1}{\sqrt{1-x_i}} - \sqrt{1-x_i}.$$

As we must get rid of the square roots in order to use the given condition, we may try to build up the following inequalities by repeated application of Cauchy's inequality together with the given constraint:

$$\left(\sum_{i=1}^{n}\sqrt{1-x_i}\right)^2 \le n(n-1),$$

$$n^2 \le \left(\sum_{i=1}^{n}\sqrt{1-x_i}\right)\sum_{i=1}^{n}\frac{1}{\sqrt{1-x_i}} \quad \text{and}$$

$$\left(\sum_{i=1}^{n}\sqrt{x_i}\right)^2 \le n.$$

Whereas the synthetic approach proceeds from given conditions and established inequalities to prove the required inequality, the analytic approach is somehow the reverse - by investigating the required inequality to find out conditions for its validity until we arrive at some established inequalities. However, in using the analytic approach, we must ensure that every step in the deduction is reversible so that each intermediate inequality is a sufficient condition for the validity of the preceeding one.

[West Germany 1987]

Let k and n be natural numbers such that $1 \le k \le n$. If $x_1, x_2, ..., x_k$ are k positive numbers with their product equal to their sum, prove that

$$x_1^{n-1} + x_2^{n-1} + ... + x_k^{n-1} \ge kn.$$

Determine a necessary and sufficient condition for the equality to hold.

Let us denote $x_1 + x_2 + ... + x_k$ and $x_1 x_2 ... x_k$ by T. By using the A.M.-G.M. inequality, we obtain

$$x_1^{n-1} + x_2^{n-1} + ... + x_k^{n-1} \ge kT^{\frac{n-1}{k}}.$$

So it suffices to prove that

$$kT^{\frac{n-1}{k}} \ge n.$$

Since it is given that the sum of the x_i's is the same as their product, we are prompted to try the A.M.-G.M. inequality once again, but on T this time:

$$\frac{T}{k} \geq T^{\frac{1}{k}}$$
$$\Leftrightarrow T^{\frac{n-1}{k}} \geq k^{\frac{n-1}{k-1}}.$$

Hence it suffices to prove that

$$k^{\frac{n-1}{k-1}} \geq n,$$

which is equivalent to

$$k \geq n^{\frac{k-1}{n-1}}.$$

This last inequality is exactly the A.M.-G.M. inequality for $(k-1)$ n's and $(n-k)$ 1's.

Global View versus Partial Attack

Some olympiad problems comprise several parts with similar properties. In such cases, we may try to attack a representative part and then generalise to a global solution. This approach is particularly useful for problems with additive features such as length, area, counting and inequality.

[China 1986]

Can we arrange two 1's, two 2's, ..., two 1986's in a sequence so that there are exactly i numbers between any two i's $(i = 1, 2, ..., 1986)$*?*

Consider the relative positions of two elements a and b (each used twice) first. Observe that in every arrangement, there is always an *even* number of elements sandwiched between another pair of identical numbers. This partial property can be generalised to all the numbers under consideration. On the other hand, since there must be exactly i numbers between any two i's, the total number of elements sandwiched between pairs of identical elements is $1 + 2 + 3 + ... + 1986$, which is *odd* and hence we obtain a contradiction.

The targets in some olympiad problems are actually parts of more general ones. Proposers of these problems may have picked out these parts for the sake of hidding some nice properties of the general situation as well as breaking the links between individual parts. Thus the problem becomes more difficult. We may try to generalize the given problem and then solve the more general problem, which is sometimes easier. Let us try the next example.

[USA 1975]
If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, ..., n$, determine $P(n+1)$.

Construct another polynomial, based on $P(x)$, with $0, 1, 2, ..., n$ as its zeros. Use factorisation and substitution to determine the leading coefficient of this polynomial. In this problem, we generalise from the narrow target to searching for the polynomial and then return to the target by simple substitution.

Recurrence Relations

The use of recurrence relations is a common and important method of solving olympiad problems with natural numbers as parameters. The crucial step in this method is the construction of recurrence relations in the form of equalities or inequalities. For some problems it may be helpful to build up new and more useful recurrence relations even though some recurrence relations are given by the problems. We illustrate two common situations each with an example. The first one is solved by constructing a recurrence relation through the general term of a sequence.

[IMO 1972]
Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Let $a_{m,n}$ be the given expression. Establish a recurrence relation among $a_{m,n}, a_{m-1,n+1}$ and $a_{m-1,n}$. Then use induction.

The next example illustrates the construction of recurrence relations through undetermined coefficients.

[Austria 1983]

A sequence $\{x_n\}$ is defined by $x_1 = 2$, $x_2 = 3$,

$$x_{2m+1} = x_{2m} + x_{2m-1} \qquad (m \geq 1)$$
$$\text{and} \quad x_{2m} = x_{2m-1} + 2x_{2m-2} \qquad (m \geq 2).$$

Determine x_n as a function of n.

Let $x_{2m+1} + ax_{2m} = b(x_{2m-1} + ax_{2m-2})$ where a and b are to be determined by substitution and comparison with the given relations. x_{2m+1} and x_{2m} can then be found by substitution and elimination.

Using the Binary System

Some olympiad problems are explicitly or implicitly related to the integer 2. The use of binary system in these problems often gives easier and simpler solutions. Let us first investigate the use of binary system in a problem on divisibility.

[Canada 1985]

Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k.

Note that if p is a prime number, then the highest power of p dividing $n!$ is

$$\sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

The 'only if' part is straightforward. For the 'if' part, let $n = (a_1 a_2 \ldots a_m)_2$ where $a_1 = 1$. Then

$$\left[\frac{n}{2^i}\right] = (a_1 a_2 \ldots a_m)_2 \quad (i = 1, 2, \ldots, m-1).$$

Since $2^{k-1} | n!$ if

$$\sum_{i=1}^{m-1} \left[\frac{n}{2^i}\right] \geq n - 1,$$

we then try to show that this inequality is equivalent to $a_2 + a_3 + \ldots + a_m \leq 0$.

Expressing numbers in base 2 may also facilitate the solutions of equations related to powers of 2 in olympiad problems.

[Canada 1981]
Show that the equation

$$[x]+[2x]+[4x]+[8x]+[16x]+[32x]=12345$$

has no real solution.

Initial investigation will lead to the bounds $195 < x < 196$. Let $\{x\} = (0.abcde...)_2$. Show that the given equation has real solution if and only if $31a + 15b + 7c + 3d + e = 60$, which is impossible since a, b, c, d and e are either 0 or 1.

The use of binary system may also help us in evaluating sums involving powers of 2.

[IMO 1968]
For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right].$$

Let $n = (a_m a_{m-1} ... a_1 a_0)_2$. Show that, for $k = 0, 1, 2, ..., m-1$,

$$\left[\frac{n+2^k}{2^{k+1}}\right] = a_m 2^{m-k-1} + a_{m-1} 2^{m-k-2} + \ldots$$
$$+ a_{k+2}2 + a_{k+1} + a_k$$

and $\left[\dfrac{n+2^m}{2^{m+1}}\right] = a_m$.

Binary numbers are also the tool for solving some problems on recurrence relations which rely on powers of 2.

[IMO 1988]
A function f is defined on the positive integers by

$$f(1)=1,\ f(3)=3,\ f(2n)=f(n),$$
$$f(4n+1)=2f(2n+1)-f(n),$$
$$f(4n+3)=3f(2n+1)-2f(n)$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, *for which* $f(n) = n$.

As the definition of $f(n)$ relies on powers of 2, tabulation of n and $f(n)$ in base 2 will uncover the pattern. The conjecture can then be proved by induction. Note that $f(n) = n$ if and only if n is a binary palindrome.

The Method of Infinite Descent

In recent olympiads, particularly at the IMO, there were problems which could be solved by the method of infinite descent. This method may be regarded as a combination of induction with proof by contradiction. The history of this method dated back to the 17th century when Fermat used it to prove that the equation $x^4 + y^4 = z^4$ has no non-trivial integral solutions. We shall use some recent IMO problems to demonstrate the different approaches in applying this method.

1. Least Element Approach

This strategy is based on the **Well-Ordering Principle:**

Every non-empty set of natural numbers has a unique least element.

[IMO 1987]
Let n be an integer greater than or equal to 2. *Prove that if* $k^2 + k + n$ *is prime for all integers n such that* $0 \le k \le \sqrt{(n/3)}$, *then* $k^2 + k + n$ *is prime for all integers k such that* $0 \le k \le n - 2$.

Let $A = \{0, 1, 2, ..., n-2\}$ and $B = \{x \in A : \sqrt{(n/3)} < x < n-2$ and $x^2 + x + n$ is a composite$\}$. The problem is equivalent to showing that B is empty. Suppose B is non-empty. Denote its least element by m. We have $m^2 + m + n = pq$ for some natural numbers p and q such that $1 < p \le q$. Using this assumption and the given conditions, we can easily deduce that $p < 2m$ and $p < n$. Let $t = |m - p| (< m)$. Then, corresponding to the cases $p \le m$ and $m < p < 2m$, we can deduce that $t^2 + t + n$ and $(t-1)^2 + (t-1) + n$ respectively belong to B and are thus composite. This contradicts the assumption that m is the least element of B.

2. Functional Value Approach

Some olympiad problems required us to prove the validity of a proposition on a set of n-tuples of natural numbers. For this class of problems, a useful strategy is to construct a suitable function from the set the n-tuples to the set of all natural numbers. The range of the function becomes a set of natural numbers for applying the least element approach. The solution, for which a special prize was awarded, submitted by a Bulgarian girl to one of the most difficult IMO problems exemplifies this strategy:

[IMO 1988]
Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Let $A = \{(a,b) : a, b \in \mathbb{N}, a \geq b \text{ and } ab+1|a^2+b^2\}$ and denote the set

$$\{(a,b) \in A : \frac{a^2+b^2}{ab+1} \text{ is not the square of an integer}\}$$

by B. If B is non-empty, then we may define a function $f : B \to \mathbb{N}$ by $f(a,b) = a+b$ and let $f(a_o, b_o)$ be the least element of $f[B]$. If q is not the square of an integer, then

$$\frac{x^2+b^2}{xb+1} = q$$

$$\Leftrightarrow \quad x^2 - qbx + b^2 - q = 0 \qquad (*)$$

Using the relations between roots and coefficients, we can deduce that, in addition to (a_o, b_o), there is another solution (a_1, b_o) of (*) such that $a_1 + b_o < a_o + b_o$. This contradicts our assumption.

3. Set Contraction Approach

In olympiad problems we are often given a finite set (A_o, say) and we are required to prove that a proposition P is true for every element of A_o. For some of these problems, there exists a **proper** subset A_1 of A_o such that if P is true for every element of A_1, then P will be true for every element of A_o. Furthermore, these problems also possess the nice property that (A_o, A_1) in the foregoing statement can be extended to (A_1, A_2) and so on. This process generates a sequence of sets:

$$A_0 \supset A_1 \supset A_2 \supset ... \supset A_n.$$

Since A_o is finite and the number of elements in the subsets in this sequence is strictly decreasing, for some A_n with only a few elements it should be easy for us to prove that P is true for every element in A_n.

[IMO 1985]
Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is coloured either blue or white. It is given that
(i) *for each $i \in M$, both i and $n - i$ have the same colour;*
(ii) *for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.*
Prove that all elements in M must have the same colour.

Denote by $x \sim y$ if x and y have the same colour. Use the given conditions to show that, for each $i \geq k$, $i \sim x$ for some $x \in M_1 = \{1, 2, ..., k-1\}$. Let r be the remainder when n is divided by k. Then r and k are also relatively prime. Verify that the given conditions still hold when n and k are replaced respectively by k and r. The process can be continued for smaller and smaller subsets $M_2, M_3, ...$ until we arrive at some $M_j = \{1, 2, ..., n_j - 1\}$ and $k_j = 1$, for which it is straightforward to prove that the proposition is true.

4. Index Function Approach

Some olympiad problems required us to prove that an operation can only be repeated finitely many times. We may correspond each round of the operation with a positive integral-valued function constructed appropriately on the variables related to the operation so that the functional value decreases with each round of the operation. Since the initial value is finite, this process cannot be continued indefinitely.

[IMO 1986]
To each vertex of a regular polygon an integer is assigned in such a way that the sum of all the five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$, then

the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Let x_1, x_2, x_3, x_4, x_5, in this order, be the five numbers at the vertices. Construct the function

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= (x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 \\ &\quad + (x_4 - x_1)^2 + (x_5 - x_2)^2. \end{aligned}$$

This pilot study revealed that, for problems at olympiad level, while heuristics suggested by Pòlya are useful in analysing the problems and in exploring feasible solutions, most of the more effective strategies are topic-oriented. Olympiad problems in geometry are almost excluded from this report because although common strategies for solving them do exist, such as *expressing quantities in terms of areas of triangles*, they are confined to geometry. Possible directions for research include studies on problem-solving strategies for individual areas of olympiad mathematics: Euclidean geometry, algebra, number theory and combinatorics, studies on the aspect of control, and studies on teaching students how to solve olympiad problems. These studies will be facilitated if more experts publish their experience in solving olympiad problems. Expert solutions are very useful but a cruicial question is: 'How can you think of such a solution?'

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