Problems and Solutions

Problems and Solutions. The aim of this section is to encourage readers to participate in the intriguing process of problem solving in Mathematics. This section publishes problems and solutions proposed by readers and editors.

Readers are welcome to submit solutions to the following problems. Your solutions, if chosen, will be published in the next issue, bearing your full name and address. A publishable solution must be correct and complete, and presented in a well-organised manner. Moreover, elegant, clear and concise solutions are preferred.

Readers are also invited to propose problems for future issues. Problems should be submitted with solutions, if any. Relevant references should be stated. Indicate with an ° if the problem is original and with an * if its solution is not available.

All problems and solutions should be typewritten double-spaced, and two copies should be sent to the Editor, Mathematical Medley, c/o Department of Mathematics, The National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

Problems

P20.1.1. *Proposed by the Editor.*

Let f be a real-valued function of a real variable whose derivatives up to 4th order are continuous on $[-\sqrt{3}, \sqrt{3}]$. If $f(\pm 1) = 0$ and $|f^{(4)}(x)| \leq 1 \ \forall\, x \in [-\sqrt{3}, \sqrt{3}]$, show that

$$\left| \int_{-\sqrt{3}}^{\sqrt{3}} f(x)dx \right| \leq \frac{\sqrt{3}}{15}.$$

Solutions

The following are solutions of the problems in Volume 19, Number 2, December 1991.

P19.1.1. *Solution by the Editor.*

Let $x_0 := 1$, and $x_n := (2x_{n-1})^{\frac{1}{4}}$, $n = 1, 2, \ldots$. Then, by induction,

$$x_n = 2^{(1+4+\ldots+4^{n-1})/4^n}$$
$$= 2^{(4^n-1)/3\times 4^n}, \quad n = 0, 1, \ldots$$

Hence $x_n \to 2^{\frac{1}{3}}$ as $n \to \infty$.

From the solution, one can easily construct similar algorithms to approximate other roots of numbers. For instance, to compute $\sqrt[5]{2}$, we let $x_0 = 1$, and $x_n = (2x_{n-1})^{\frac{1}{6}}$, $n = 1, 2, \ldots$. This means that each step we multiply the previous number by 2 and take square root and cube root successively. Then

$$x_n = 2^{(6^n-1)/5\times 6^n} \to 2^{\frac{1}{5}}.$$

P19.2.1. *Official Solution.*

Using the sine rule in the triangles ΔAIC and $\Delta AIC'$ gives $IC'/CI = AC'/b$ and similarly considering the triangles ΔCIB and $\Delta BIC'$ gives $IC'/CI = C'B/a$. Hence

$$c = AC' + C'B = (a+b)\frac{IC'}{CI}$$

which implies

$$\frac{CI}{CC'} = \frac{a+b}{a+b+c}.$$

Analogously we get

$$\frac{AI}{AA'} = \frac{b+c}{a+b+c} \quad \text{and} \quad \frac{BI}{BB'} = \frac{a+c}{a+b+c}.$$

The inequality between the arithmetic and geometric means gives

$$\frac{1}{3}\left(\frac{b+c}{a+b+c} + \frac{a+c}{a+b+c} + \frac{a+b}{a+b+c}\right) \geq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}}.$$

Hence

$$\left(\frac{2}{3}\right)^3 \geq \frac{AI \cdot BI \cdot CI}{AA'BB'CC'}$$

and we have proved the right inequality. To prove the left inequality we recall the following simple facts:

1. If $x + y = x_1 + y_1$ and $|x - y| < |x_1 - y_1|$ then $x^3 + y^3 < x_1^3 + y_1^3$ for positive real numbers.
2. For any real numbers a, b, c the following equality holds

$$3(a+b)(b+c)(c+a) = (a+b+c)^3 - a^3 - b^3 - c^3.$$

Suppose that $a \geq b \geq c$. Using the triangle inequality and $c > 0$ we get $(a+b+c)/2-(a+b-c)/2 = c > |a-b+$ and $(a+b+c)/2 > |(a+b-c)/2-c|$. Thus

$$\begin{aligned}\frac{AI \cdot BI \cdot CI}{AA'BB'CC'} &= \frac{(a+b+c)^3 - a^3 - b^3 - c^3}{3(a+b+c)^3} \\ &> \frac{(a+b+c)^3 - ((a+b+c)/2)^3 - ((a+b-c)/2)^3 - c^3}{3(a+b_c)^3} \\ &> \frac{(a+b+c)^3 - ((a+b+c)/2)^3 - ((a+b+c)/2)^3 - c^3}{3(a+b+b)^3} \\ &= \frac{1}{4}.\end{aligned}$$

P19.2.2. *Official Solution.*

We have $a_1 = 1$ and $a_2 = p$, where p is the smallest prime number which does not divide n; $a_k = n-1$ and the constant difference $a_j - a_{j-1}$ between two consecutive terms is denoted $r = p - 1$.

If n is odd then $a_2 = 2$, $r = 1$ and the sequence is $1, 2, ..., n-1$. Since all positive integers less than n are relatively prime with n, it follows that n is a prime number.

If n is even we must have $p \geq 3$. In case $p = 3$ then $r = 2$ and the sequence is $1, 3, 5, ..., n-1$. Since for every odd $q, q < n$, we have that q and n are relatively prime, n cannot have any odd prime factors, and we deduce that $n = 2^m$ for some positive integer m.

In case $p > 3$ it follow that $3|n$. Since $a_k = a_1 + r(k-1)$, we have $n-1 = 1+(p-1)(k-1)$ which implies

$$(p-1)|(n-2). \tag{1}$$

Let q be a prime number such that $q|(p-1)$. From (1) we deduce that $q|(n-2)$. Now $q < p$ and thus $q|n$. Since also $q|(n-2)$ it follows that $q|2$ and consequently $q = 2$. Hence all prime factors of $p-1$ are equal to 2, and it follows that $p-1 = 2^\ell$, or $p = 2^\ell + 1$, where $\ell \geq 2$. Note that for any integer $j \geq 0$, $3|(2^{2j+1}+1)$. Thus since p is a prime number ℓ must be even, $\ell = 2j$. We have $a_3 = a_1 + 2r = 1+2(p-1) = 2p-1 = 2^{2j+1}+1$ which is divisible by 3. But $3|a_3$ and $3|n$ contradict the fact that a_3 and n are relatively prime. The proof is complete.

P19.2.3. *Official Solution.*

Set $A_1 = \{k \in S; 2|k\}$, $A_2 = \{k \in S; 3|k\}$, $A_3 = \{k \in S; 5|k\}$, $A_4 = \{k \in S; 7|k\}$ and $A = A_1 \cup A_2 \cup A_3 \cup A_4$. We have $|A_1| = 140$, $|A_2| = 93$, $|A_3| = 56$ and $|A_4| = 40$. Similarly, $|A_1 \cap A_2| = 46$, $|A_1 \cap A_3| = 28$, $|A_1 \cap A_4| = 20$, $|A_2 \cap A_3| = 18$, $|A_2 \cap A_4| = 13$, $|A_3 \cap A_4| = 8$, $|A_1 \cap A_2 \cap A_3| = 9$, $|A_1 \cap A_2 \cap A_4| = 6$, $|A_1 \cap A_3 \cap A_4| = 4$, $|A_2 \cap A_3 \cap A_4| = 2$, $|A_1 \cap A_2 \cap A_4| = 1$. By the inclusion and exclusion principle

$$\begin{aligned} |A| &= |A_1 \cup A_2 \cup A_3 \cup A_4| \\ &= 140+93+56+40-46-28-20-18-13-8+9+6+4+2-1 \\ &= 216. \end{aligned}$$

For any five members in A, by the pigeonhole principle, there are two of them belonging to some A_i, $1 \leq i \leq 4$, hence these numbers are not prime to each other. We have proved that $n > 216$.

On the other hand, let $B_1 = A \setminus \{2,3,5,7\}$, $B_2 = \{11^2, 11 \times 13, 11 \times 17, 11 \times 19, 11 \times 23, 13^2, 13 \times 17, 13 \times 19\}$ and $P = S \setminus \{B_1 \cup B_2\}$. We see that $|P| = |S| - |B_1| - |B_2| = 60$ and P consists of 1 and all the prime numbers in S. Suppose that T is a subset of S with $|T| = 217$. We shall show that T contains 5 numbers which are pairwise relatively prime. It is obvious that we have only to consider the case when $|T \cap P| \leq 4$. In this case $|T \cap (S \setminus P)| \geq 217 - 4 = 213$, i.e., among all composite numbers in S (total number is $|S \setminus P| = 220$), there are at most 7 members which are not in T. Let $M_1, M_2, ..., M_8$ denote the sets

$$\{2 \times 23, 3 \times 19, 5 \times 17, 7 \times 13, 11 \times 11\},$$
$$\{2 \times 29, 3 \times 23, 5 \times 19, 7 \times 17, 11 \times 13\},$$
$$\{2 \times 31, 3 \times 29, 5 \times 23, 7 \times 19, 11 \times 17\},$$
$$\{2 \times 37, 3 \times 31, 5 \times 29, 7 \times 23, 11 \times 19\},$$
$$\{2 \times 41, 3 \times 37, 5 \times 31, 7 \times 29, 11 \times 23\},$$
$$\{2 \times 43, 3 \times 41, 5 \times 37, 7 \times 31, 13 \times 17\},$$
$$\{2 \times 47, 3 \times 43, 5 \times 41, 7 \times 37, 13 \times 19\},$$
$$\{2^2, 3^2, 5^2, 7^2, 13^2\},$$

respectively. Obviously $M_i \subset S \setminus P$, $i = 1, 2, ..., 8$. By the pigeonhole principle there is an i_0, $1 \leq i_0 \leq 8$, such that $M_{i_0} \subset T$. The five members of M_{i_0} are pairwise relatively prime. Summing up, we conclude that $n = 217$.

P19.2.4. *Official Solution.*

Start at some vertex v_0. Walk along distinct edges of the graph, numbering them $1, 2, ...$ in the order you encounter them, until it is no longer possible to proceed without reusing an edge.

If there are still edges which are not numbered, one of them has a vertex which has been visited, for else G would not be connected. Starting from this vertex, continue to walk along unused edges, resuming the numbering where you left off. Eventually you will get stuck. Repeat the procedure just described until all edges are numbered.

Let v be a vertex which is incident with e edges, where $e \geq 2$. If $v = v_0$ then v is on edge 1, so the GCD at v is 1. If $v \neq v_0$, suppose the first time you reached v was at the end of edge r. At that time there were $e - 1 \geq 1$ unused edges incident with v, so one of them was labeled $r + 1$. The GCD of any set containing r and $r + 1$ is 1.

P19.2.5.

We denote $\alpha = \angle PAB$, $\beta = \angle PBC$, $\gamma = \angle PCA$ and $x = AP$, $y = BP$, $z = CP$.

Assume that α, β, γ are all greater than $30°$. We want to prove that this leads to a contradiction. Clearly the angles $\angle A, \angle B, \angle C$ are $< 120°$. Let

T be the area of the triangle ΔABC.

$$\begin{aligned} T &= \frac{1}{2}cx \sin \alpha + \frac{1}{2}ay \sin \beta + \frac{1}{2}bz \sin \gamma \\ &> \frac{1}{4}(cx + ay + bz) \end{aligned}$$

where we have used that $\sin\alpha, \sin\beta, \sin\gamma$ are all $> 1/2$. As will be proved below $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$ and hence

$$a^2 + b^2 + c^2 \geq \sqrt{3}(cx + ay + bz). \tag{1}$$

By the cosine theorem

$$\begin{aligned} x^2 &= b^2 + z^2 - 2bz \cos \gamma > b^2 + z^2 - \sqrt{3}bz \\ y^2 &= c^2 + x^2 - 2cx \cos \alpha > c^2 + x^2 - \sqrt{3}cx \\ z^2 &= a^2 + y^2 - 2ay \cos \beta > a^2 + y^2 - \sqrt{3}bz \end{aligned} \tag{2}$$

using $\cos \alpha, \cos \beta, \cos \gamma < \sqrt{3}/2$. Adding the inequalities in (2) we get

$$a^2 + b^2 + c^2 < \sqrt{3}(cx + ay + bz),$$

which contradicts (1).

It remains to prove the inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$. Using the cosine law and the area law we see that

$$a^2 + b^2 + c^2 = 4T(\cot \alpha + \cot \beta + \cot\gamma)$$

and hence we have to prove that

$$\cot \alpha + \cot \beta + \cot \gamma \geq \sqrt{3}. \tag{3}$$

If α, β, γ all lie in $(0, \pi/2]$ we use the convexity of $u \mapsto \cot u$; in the case when for example $\gamma \in (\pi/2, \pi)$ we can write

$$\begin{aligned} \cot \alpha + \cot \beta + \cot \gamma &\geq 2\cot \frac{\alpha + \beta}{2} + \cot \gamma \\ &\geq \cot \frac{\alpha + \beta}{2} - \cot(\alpha + \beta) \end{aligned}$$

which gives

$$\cot\alpha + \cot\beta + \cot\gamma \geq 2c - \frac{c^2-1}{2c}$$
$$\geq \frac{3}{2}\,c + \frac{1}{2c} \geq \sqrt{3}$$

with $c = \cot\dfrac{\alpha+\beta}{2}$. This proves (3) and completes the proof.

P19.2.6. *Official Solution.*

Every natural number can be expressed (uniquely) in a binary representation. (This binary representation makes the notation more readable.) Suppose $i = b_0 + b_1 2 + b_2 2^2 + \ldots + b_k 2^k$ with all $b_i \in \{0,1\}$ and $k \in \mathbf{N}$. Define

$$h_i = b_0 + b_1 2^{-a} + b_2 2^{-2a} + \ldots + b_k 2^{-ka}$$

so that

$$0 \leq h_i \leq 1 + 2^{-a} + 2^{-2a} + \ldots + 2^{-ka} \leq \frac{1}{1-2^{-a}}.$$

Take $j, j \neq i$, $j = c_0 + c_1 2 + \ldots + c_k 2^k$, $c_i \in \{0,1\}$ and let $t = \min\{p \in \{0,1,2,\ldots,k\} | b_p \neq c_p\}$. Then

$$|h_i - h_j| = |(b_0 - c_0) + (b_1 - c_1)2^{-a} + (b_2 - c_2)2^{-2a} + \ldots + (b_k - c_k)2^{-2ka}|$$
$$\geq |(b_t - c_t)2^{-ta}| - |(b_{t+1} - c_{t+1})2^{-(t+1)a}| - \ldots |(b_k - c_k)2^{-ka}|$$
$$\geq 2^{-ta} - \frac{2^{-(t+1)a}}{1-2^{-a}} = 2^{-ta}\left(1 - \frac{2^{-a}}{1-2^{-a}}\right) = (2^t)^{-a}\left(\frac{2^a-2}{2^a-1}\right)$$
$$\geq \left(\frac{2^a-2}{2^a-1}\right)|i-j|^{-a}$$

The last inequality used the fact that $2^t \leq |i-j|$. To see this put $s = \max\{p \in \{0,1,\ldots,k\} | b_p \neq c_p\}$. Then

$$|i-j| \geq |(b_s - c_s)2^s| - \sum_{k=t}^{s-1} 2^k = 2^t.$$

So

$$|h_i - h_j| \cdot |i-j|^{-a} \geq \frac{2^a-2}{2^a-1}.$$

Define $x_i = \dfrac{2^a-2}{2^a-1} \cdot h_i$.