The Icosahedron*

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§1. Introduction

Let me first offer my congratulations to all the prize-winners of the year's competition. I hope that all of you will pursue your mathematical studies, either for their own sake or as a basis for other disciplines. Mathematics lies at the heart of our technological society and it needs to recruit youthful talent in each generation.

Turning to the subject of my lectures, let me explain why I have chosen it. Mathematics consists on the one hand of general theories, large structures with elaborate machinery and having wide applicability, and on the other hand of very special exceptional items. The interplay between the general and the exceptional provides much of the beauty and attraction of mathematics. Special results develop into general theories and these theories in turn cast light on new problems.

The icosahedron, and the other regular solids, have been known since the time of the ancient Greeks and have played an important part in the development of mathematics. Moreover they turn out to have unexpected relations to many other topics in mathematics, some of which are at the frontier of current research. Unfortunately, shortage of time prevents me from a full survey, including recent developments, but I hope I will nevertheless be able to convey some of the beauty and fascination associated with the icosahedron.

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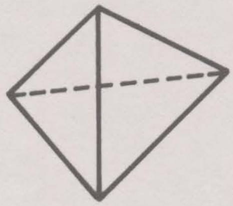
§2. The regular solids

Let me begin by rapidly reviewing all the five regular solids. The simplest is the *tetrahedron*



which has four triangular faces, four vertices and six edges, or symbolically

$$F = 4, \quad E = 6, \quad V = 4.$$

Note that the tetrahedron is self-dual, i.e. the figure whose vertices are the mid-points of the faces of the tetrahedron is again a regular tetrahedron.

The *symmetries* of a regular solid are the motions (rotations about the centre) which transform it into itself but permute the vertices. Since any vertex can be moved to any other vertex, and subsequently the three edges through this vertex can be rotated, we get in all

$$4 \times 3 = 12 = \frac{1}{2}(4!)$$

symmetries.

Next consider the *cube*, for which

$$F = 6, \quad E = 12, \quad V = 8.$$

Its dual is the *octahedron* for which F and V are interchanged,

$$F = 8, \quad E = 12, \quad V = 6.$$

The symmetries of the cube and octahedron coincide and can be computed from either figure, giving

$$8 \times 3 = 6 \times 4 = 24 = 4!.$$

Finally we come to the subject of the lecture, the *icosahedron* and its dual the *dodecahedron.* The icosahedron has 20 triangular faces and

$$F = 20, \quad E = 30, \quad V = 12$$

whereas the dodecahedron has 12 pentagonal faces, so that

$$F = 12, \quad E = 30, \quad V = 20.$$

Note that Euler's famous formula

$$F - E + V = 2$$

gives a useful check on the numbers involved.

The number of symmetries is

$$20 \times 3 = 12 \times 5 = 60 = \frac{1}{2}(5!).$$

In addition to the regular solids we should also consider the regular polygons, the 2-dimensional counterparts. Clearly a polygon with n sides has just n symmetries.

So far I have just enumerated the *number* of symmetries of a given regular solid. However, this number is an inadequate reflection of the subtlety of the situation. For example a polygon with 60 vertices will have the same *number* of symmetries as an icosahedron but the symmetries themselves are differently organized. The key fact is that any 2 symmetries a and b can be composed (performed one after the other) to get a new symmetry which is called their *product* and denoted simply by ab. Note that the order here is important and in general,

$$ab \neq ba.$$

The symmetries of a regular solid are an example of the mathematical notion of a *group*. Groups are essentially the mathematical abstraction of the notion of symmetry and they are one of the most important ideas in the whole of mathematics.

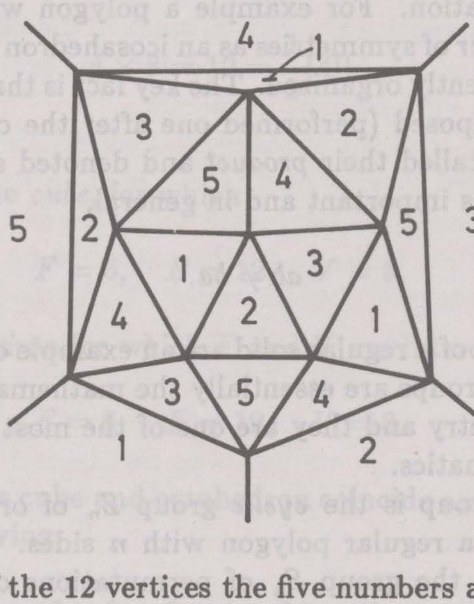
The simplest group is the *cyclic* group Z_n of order n. This is the symmetry group of a regular polygon with n sides. A more important example is given by the group S_n of permutations of n objects. S_n is called the *symmetric* group and has $n!$ elements. Also important is the *alternating* group A_n consisting of even permutations in S_n. It has $\frac{1}{2}(n!)$ elements.

We can now try to identify the groups of symmetries of the regular solids. For the tetrahedron it is easy to see that we get the group A_4 of even permutations of the 4 vertices. For the cube we get the symmetric group S_4 of permutations of the 4 main diagonals.

For the icosahedron the numbers suggest that the symmetry group, having order $\frac{1}{2}(5!)$, should be A_5. We need to find 5 objects which are permuted by its symmetries. This is considerably more difficult to see than for the tetrahedron or cube, and the five objects are more intricate than vertices or diagonals. In fact it is possible to *colour* the 20 faces of the icosahedron with five different colours, each colour being used four times, in a symmetrical manner. Any symmetry of the icosahedron will then permute the five colours and give rise to the group A_5 of all even permutations. The colouring of the icosahedron may be taken as illustrated below, where a planar diagrammatic scheme for the icosahedron has been employed, and numbers 1, 2, 3, 4, 5 are used instead of colours. The colouring (or numbering) rule may be stated as follows :

cross over, turn left, then right.

Starting from a triangle numbered 1 we perform these moves to get to other triangles numbered 1. A similar procedure applies to the other numbers.



Note that at each of the 12 vertices the five numbers appear in a different cyclic order.

§3. Solving equations by radicals

As is well-known a general quadratic equation

$$ax^2 + bx + c = 0$$

can be solved explicitly by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Similar, but more complicated, expressions exist for solving cubic and quartic equations. The essential point is that these expressions involve cube roots and square roots. However, it was the great achievement of Galois to show that the general *quintic* equation cannot be solved by using "radicals", i.e. square roots, cube roots, etc. In fact Galois invented group theory for precisely this purpose, the relevant symmetry being that between the n roots of an equation of degree n. The ultimate reason why equations of degree n can be solved by radicals for $n = 2, 3, 4$ and not for $n \geq 5$ is that *the symmetric group S_n can, for $n \leq 4$, be decomposed into cyclic groups* while this is not true for $n \geq 5$.

The decomposition of S_n for $n = 2, 3, 4$ can easily be described. For $n = 2$, $S_2 = Z_2$ is itself cyclic of order 2. For $n = 3$, the alternating group $A_3 = Z_3$ is cyclic of order 3 and the sign of a permutation maps $S_3 \to \{\pm 1\}$ i.e. to Z_2. Thus cubics can be solved by using a cube root and a square root. For $n = 4$, we have to study the group A_4 in more detail. As we saw in §2, A_4 is the symmetry group of the tetrahedron. It therefore permutes the three pairs of opposite edges, and gives rise to the cyclic group Z_3. Elements of A_4 which have no effect on opposite pairs must then interchange two pairs of vertices, e.g.

$$1 \leftrightarrow 2 \quad \text{and} \quad 3 \leftrightarrow 4.$$

This shows that A_4 decomposes into Z_3 and two copies of Z_2. Finally this means a quartic equation can be solved by using one cube root and three square roots.

This procedure turns out to stop at $n = 5$ because the group A_5 cannot be decomposed (it is called a *simple* group). Recall that A_5 is the symmetry group of an icosahedron. Thus the indecomposability of A_5, a reflection of the subtle symmetry structure of the icosahedron, explains the insolubility of quintic equations. This unexpected abstract connection between the icosahedron and quintic equations is a magnificent example of the power of mathematical ideas.

§4. Continuous groups

The groups we have encountered so far are *finite* groups. However infinite "continuous" groups also occur naturally as symmetries. The symmetry groups of a circle or a sphere are obviously infinite since they consist of all rotations in the plane or in three dimensions. They are called continuous groups because they depend on continuous parameters, the angle (and axis) of rotation. They were studied systematically by the Norwegian mathematician Sophus Lie and are now called Lie groups.

Algebraically we get examples of Lie groups from linear transformations. For example for two variables the transformations

$$y_1 = ax_1 + bx_2$$
$$y_2 = cx_1 + dx_2$$

with $ad - bc \neq 0$ form the general linear group GL_2. In matrix notation we write $y = Ax$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

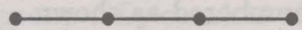
is a matrix with non-zero determinant. The coefficients a, b, c, d may be real or complex numbers. If we insist that $ad - bc = 1$ then we get the *special* linear group SL_2. All this generalizes to n variables and gives the groups GL_n and SL_n.

Lie groups turn out, in many ways, to be easier than finite groups. In particular the simple Lie groups have long been completely classified (whereas for finite simple groups the classification has only recently been completed, with much greater difficulty). For simple Lie groups it turns out that besides the obvious families of linear groups, such as SL_n, there are just five exceptional Lie groups. These are labelled

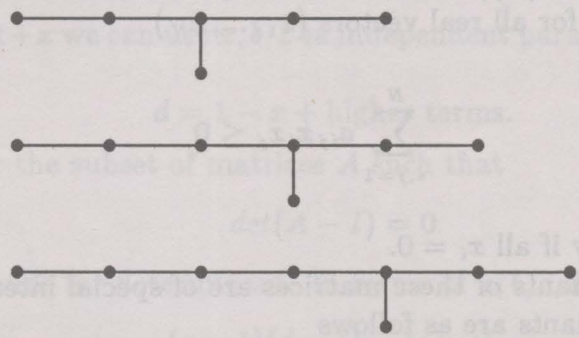
$$G_2,\ F_4,\ E_6,\ E_7,\ E_8,$$

and the last three (the E series) turn out to have a remarkable and mysterious relation to the finite symmetry groups of the regular solids. In particular the last exceptional group E_8 is related to the icosahedron.

I shall try to explain this relation in terms of the diagrams which are used in the classification of Lie groups. For example the diagram for SL_5 is just



while for SL_n it contains $(n-1)$ vertices or nodes. For E_6, E_7, E_8 the diagrams are



with 6, 7, 8 nodes respectively.

Now to each such diagram we can associate a symmetric matrix of integers where the size N of the matrix is given by the number of nodes and the entries of the matrix are given by the following rules.

(i) number the nodes of the diagram from 1 to N,
(ii) put -2 on the diagonal of the matrix,
(iii) put 1 in the (i,j) entry if nodes i and j are adjacent in the diagram,
(iv) put 0 elsewhere.

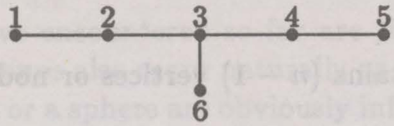
For example for SL_5 the matrix is

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

while for E_6 it is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

where the nodes in the diagram are numbered as shown



These matrices (a_{ij}) all have the key property that they are ***negative definite***, i.e. that for all real vectors $(x_1, \ldots, x_N)$

$$\sum_{i,j=1}^{N} a_{ij} x_i x_j \leq 0$$

with equality only if all $x_i = 0$.

The determinants of these matrices are of special interest. One finds that the determinants are as follows

SL_n	$(-1)^{n-1}n$
E_6	3
E_7	-2
E_8	1

The numbers 2, 3 in this table are related to the solubility of quartic equations (recall that the symmetry groups of the tetrahedron and cube are A_4 and S_4 respectively). The fact that the determinant of the E_8 matrix is 1 is related to the simplicity of A_5, the icosahedral group. In fact the E_8 matrix is a very remarkable matrix. If we look for an $N \times N$ symmetric matrix of integers which is negative definite and has determinant 1 then one can show that N must be divisible by 8 and the E_8 matrix is (up to a change of basis) the only 8×8 such matrix.

§5. From Lie groups to diagrams

In the last section I mentioned that each simple Lie group had a diagram and hence an associated matrix. I now want to indicate one way in which these diagrams arise from the Lie groups. I will start with the basic example of SL_2, although the diagram in that case is rather trivial, consisting of just one isolated node.

The Lie group SL_2 consists of 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with } ad - bc = 1.$$

Let us examine A near the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Putting $a = 1 + x$ we can use x, b, c as independent parameters and, solving for d, we get

$$d = 1 - x + \text{higher terms}.$$

Now consider the subset of matrices A such that

(1) $$det(A - I) = 0$$

where det stands for determinant. In terms of a, b, c, d we get

$$(a-1)(d-1) - bc = 0.$$

Substituting for a, d in terms of x we get

$$-x^2 - bc + \text{higher terms} = 0.$$

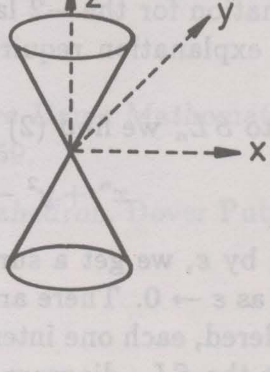
Neglecting the higher terms and putting

$$b = y + z, \quad c = y - z$$

we get the equation

(2) $$x^2 + y^2 - z^2 = 0.$$

This represents a cone in (x, y, z) space



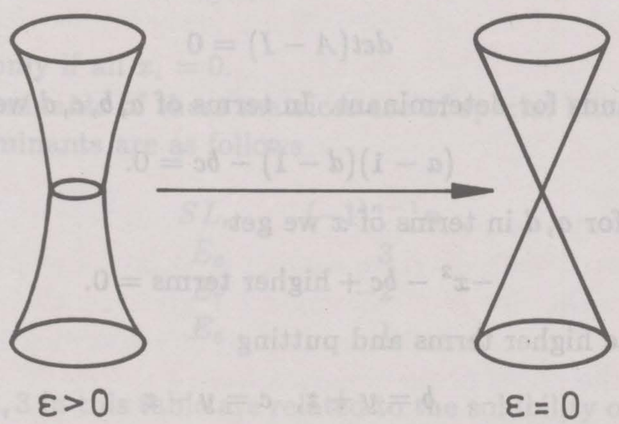
Now let us consider a small perturbation of our equation (1), replacing it by

$$\text{(3)} \qquad det(A-(1+\varepsilon)I)=0$$

where ε is small. Instead of (2) we then get

$$\text{(4)} \qquad x^2+y^2-z^2=\varepsilon^2$$

which represents a hyperboloid. As $\varepsilon \to 0$ the neck of the hyperboloid shrinks to the vertex of the cone as indicated below



The curve on the hyperboloid which shrinks to a point as $\varepsilon \to 0$ is a conic section and two conic sections intersect in two points (possibly complex). For this reason our diagram consists of a single node, labelled with -2.

Note. My explanation for the -2 labelling above is somewhat oversimplified. A proper explanation requires us to consider x, y, z as complex variables.

Generalizing to SL_n we find (2) replaced by

$$\text{(5)} \qquad x^n+y^2-z^2=0.$$

Again, perturbing by ε, we get a surface on which a collection of curves shrinks to a point as $\varepsilon \to 0$. There are $n-1$ curves in this collection and, where suitably ordered, each one intersects the next one in just one point. This is how we get the SL_n diagram of §4.

For E_8 the equation replacing (5) is

$$x^2 + y^3 + z^5 = 0 \tag{6}$$

and there are now eight curves whose intersection pattern leads to the E_8 diagram of §4.

Note that the exponents 2, 3, 5 in equation (6) are just the numbers

$$\frac{60}{E}, \quad \frac{60}{F}, \quad \frac{60}{V}$$

where E, F, V are the numbers of edges, faces and vertices of an icosahedron. This is not an accident and, while time prevents a full explanation, it should be taken as evidence that equation (6) is naturally associated with the icosahedron.

In a similar way the equations associated to E_6 and E_7 are

$$x^2 + y^3 + z^4 = 0$$
$$x^2 + y^3 + yz^3 = 0.$$

There are many further ways in which E_8 and the icosahedron enter into various parts of mathematics, but I hope the sample I have given is already an indication of these fascinating ideas.

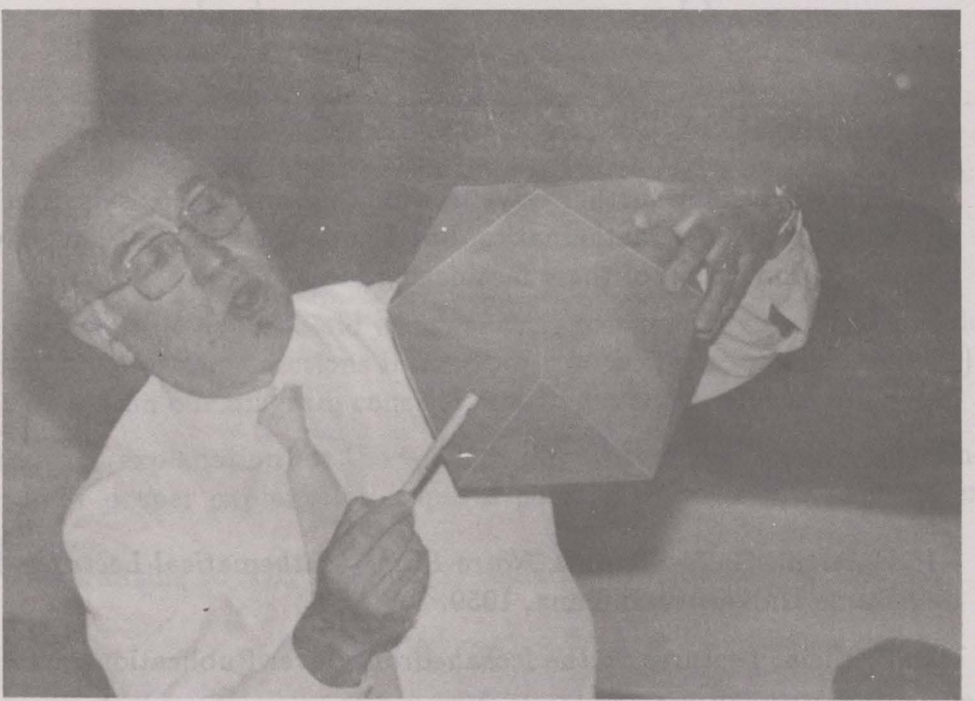
References

[1] E. Artin, Galois Theory, *Notre Dame Mathematical Lectures*, Notre Dame University, Indiana, 1959.

[2] F. Klein, *Lectures on the Icosahedron*, Dover Publications, New York 1956.

Biographical Note: Professor Atiyah was born in London in 1929 and received his B.A. in 1952 and Ph.D. in 1955 from the University of Cambridge. He was Savilian Professor of Geometry at the University of Oxford from 1963 to 1969, and a Professor and Permanent Member of the Institute for Advanced Study in Princeton from 1969 to 1972. Since 1973, he has been Royal Society Research Professor at Oxford. In October 1990 he will move to Cambridge as Master of Trinity College.

He was elected Fellow of the Royal Society in 1962 and knighted in 1984. In 1966, he was awarded a Fields Medal (the mathematical equivalent of a Nobel Prize) for his deep contributions to K-theory, the Atiyah-Singer Index theorem and the Lefschetz Fixed-Point Theorem. Professor Atiyah's work has always emphasized the unity of mathematics and is a prime example of the cross-fertilization of ideas in topology, analysis, geometry and mathematical physics. He has recently made significant contributions to the study of Yang-Mills fields. In 1986, his student, Simon Donaldson, was awarded a Fields Medal. Professor Atiyah has visited Singapore as a Lee Kuan Yew Distinguished Visitor.



"I cause a-hidden"