Singapore's participation in the 30th IMO

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The 30th International Mathematical Olympiad (IMO) was held in Brunswick (Braunschweig) of the Federal Republic of Germany from July 13 to 24, 1989. Brunswick, though a small city, is the hometown of two great German mathematicians, Carl Friedrich Gauss (1777-1855) and Richard Dedekind (1831-1916)

There was a record number of 291 contestants from 50 countries taking part in this competition. India, Portugal and Thailand were the three countries that made their first appearance at an IMO. Also Denmark and Japan, intending to take part in the next IMO, sent their observers.

The Singapore national team consisted of the team leader Prof Koh Khee Meng, the deputy leader Dr Tay Tiong Seng, and six team members, Lam Vui Chap, Lee Mun Yew, Ng Lup Keen, Tang Hsao Kun, Yeoh Yong Yeow and Yu Chang Kai.

The contest took place on July 18 and 19. There was a 4.5 hour paper consisting of 3 problems each day. The six problems and their solutions by our team members are reprinted in this issue starting from page 81. Each problem carries 7 points, and so the maximum score is 42 for each individual and 252 (=42 × 6) for each team. Medals are awarded to individuals with scores in the following ranges:

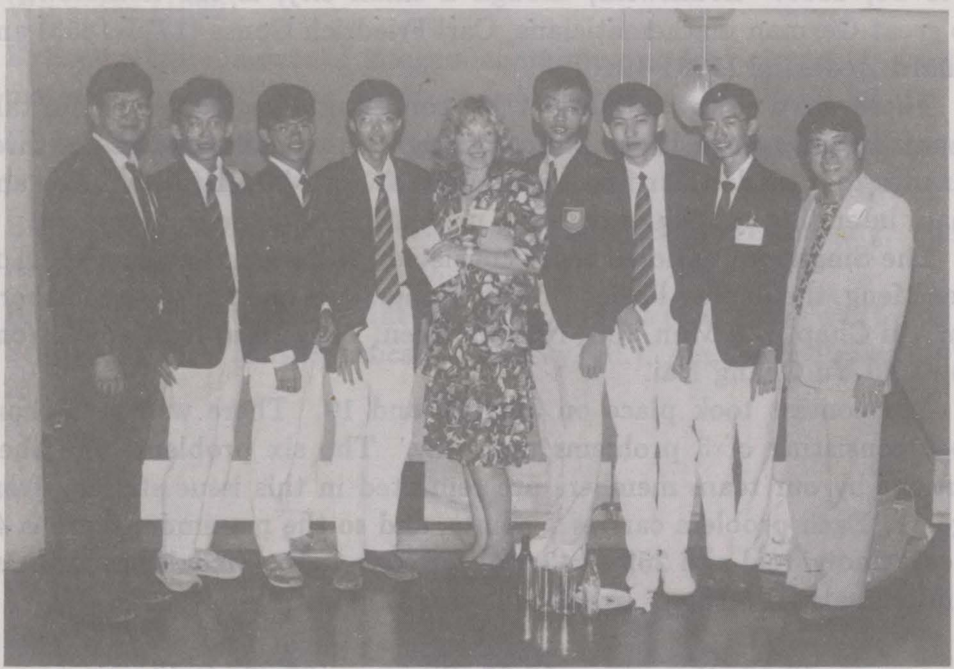
18-19: Bronze 30-37: Silver 38-42: Gold

Honourable mentions (H.M.) are awarded to contestants who obtain a perfect score of 7 for at least one of the problems.

This was the second time Singapore took part in an IMO. Last year in Australia, Singapore won two silver and two bronze medals. This year we won four bronze medals and two Honourable mentions (see table on page 81.)

It was a pity that both Yeoh Yong Yeow and Tang Hsao Kun missed a silver medal by one point each, while Lam Vui Chap and Ng Lup Keen

by one and two points, respectively. Nevertheless, the total of 143 points scored by our team ranked us 15th among the 50 participating teams and first among participating commonwealth countries (see table on page 88). This was a significant improvement on the rank, 18th among 49 participating countries, that we achieved last year. Our students had put up their best performance and we are very proud of their achievements.



The Performance of Singapore Team Members

Name	Problems						Total	Award
	1	2	3	4	5	6		
Lam V C	7	3	0	0	7	0	17	H.M.
Lee M Y	2	6	0	7	5	7	27	Bronze
Ng L K	2	7	0	7	0	0	16	H.M.
Tang H K	7	7	1	7	7	0	29	Bronze
Yeoh Y Y	1	7	6	7	6	2	29	Bronze
Yu C K	6	0	0	7	7	5	25	Bronze

Medals are awarded to individuals with scores in the following ranges:

18-19: Bronze   30-37: Silver   38-42: Gold

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30th IMO Problems

FIRST DAY

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i $(i = 1, 2, \ldots, 117)$ such that
   (i) each A_i contains 17 elements;
   (ii) the sum of all the elements in each A_i is the same.
2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_o be the point of intersection of the line AA_1 with the external bisectors of the angles B and C. Points B_0 and C_0 are defined similarly. Prove that
   (i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;
   (ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that
   (i) no three points of S are collinear, and
   (ii) for every point P of S there are at least k points of S equidistant from P.

   Prove that
$$k < \frac{1}{2} + \sqrt{2n}.$$

Time: 4.5 hours

Each Problem is worth 7 points

SECOND DAY

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that
$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. Prove that for each positive integer n there exists n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

Time: 4.5 hours

Each Problem is worth 7 points

Solutions given by our students

These have been rewritten in this presentation.

Problem 1 (solution by H K Tang): First we prove that the set $\{1,2,\ldots,351 = 3\times 117\}$ can be partitioned into 117 sets with constant sum. The following 117 sets form such a partition.

$$A_r = \{r, r+292, 236-2r\}, \qquad r = 1,2,\ldots,59$$
$$B_r = \{118-r, 293-r, 117+2r\}, \quad r = 1,2,\ldots,58$$

Next we partition the remaining numbers into 117 sets of 14 elements with constant sum. The columns of the following 14×117 array form such a partition.

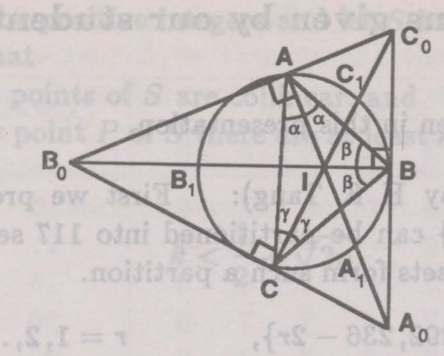
$$\begin{array}{ccccc}
352 & 353 & \cdots & 467 & 468 \\
469 & 470 & \cdots & 584 & 585 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1054 & 1055 & \cdots & 1169 & 1170 \\
1287 & 1286 & \cdots & 1172 & 1171 \\
1404 & 1403 & \cdots & 1289 & 1288 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1989 & 1988 & \cdots & 1874 & 1873
\end{array}$$

The required partition can then be formed by taking the unions of a set in the former partition with a set in the latter partition.

Problem 2 (i) (solution by H K Tang): Let I be the incentre of ΔABC. The internal and external bisectors of any angle are perpendicular to each other. Thus $AA_0 \perp B_0C_0$, $BB_0 \perp C_0A_0$, $CC_0 \perp A_0B_0$. Hence ΔABC is the orthic triangle of $\Delta A_0B_0C_0$, and I is the orthocentre of $\Delta A_0B_0C_0$. By the 9-*Point Circle Theorem*, the circumcircle of ΔABC passes through the mid-points of A_0I, B_0I and C_0I. Hence A_1, B_1 and C_1 are these mid-points. For any triangle XYZ denote its area by (XYZ). We have

$$(IA_1B) = \frac{1}{2}(A_0IB), \quad (IA_1C) = \frac{1}{2}(A_0IC), \quad (IB_1C) = \frac{1}{2}(B_0IC),$$
$$(IB_1A) = \frac{1}{2}(B_0IA), \quad (IC_1A) = \frac{1}{2}(C_0IA), \quad (IC_1B) = \frac{1}{2}(C_0IB).$$

Adding up these equations yields the desired result.



(ii) (solution by L K Ng): From (i) it suffices to prove that $(A_1B_1C_1) \geq (ABC)$. From the diagram $\angle A = 2\alpha$, $\angle B = 2\beta$, $\angle C = 2\gamma$. Since A_1, B_1 and C_1 are on the circumference, $\angle A_1 = \angle C_1A_1A + \angle B_1A_1A = \angle C_1CA + \angle B_1BA = \gamma + \beta$. Similarly, $\angle B_1 = \gamma + \alpha$, and $\angle C_1 = \alpha + \beta$. Let R be the circumradius. Then $BC = 2R\sin 2\alpha$, $AC = 2R\sin 2\beta$. Now $(ABC) = \frac{1}{2}AC \cdot BC \sin C = 2R^2 \sin 2\alpha \sin 2\beta \sin 2\gamma$. Also $B_1C_1 = 2R\sin(\gamma+\beta)$, $A_1C_1 = 2R\sin(\gamma+\alpha)$. Therefore

$$\begin{aligned}
(A_1B_1C_1) &= \frac{1}{2}B_1C_1 \cdot A_1C_1 \sin C_1 \\
&= 2R^2 \sin(\gamma+\beta)\sin(\gamma+\alpha)\sin(\alpha+\beta) \\
&= 2R^2(\sin\gamma\cos\beta + \sin\beta\cos\gamma)(\sin\gamma\cos\alpha \\
&\quad + \sin\alpha\cos\gamma)(\sin\alpha\cos\beta + \sin\beta\cos\alpha) \\
&= 2R^2(\sin^2\gamma\cos^2\beta\cos\alpha\sin\alpha + \sin^2\gamma\cos\beta\cos^2\alpha\sin\beta \\
&\quad + \sin\gamma\cos^2\beta\cos\gamma\sin^2\alpha + \sin\gamma\cos\beta\sin\alpha\cos\gamma\cos\alpha\sin\beta \\
&\quad + \sin\beta\cos\gamma\sin\gamma\cos\alpha\sin\alpha\cos\beta + \sin^2\beta\cos\gamma\sin\gamma\cos^2\alpha \\
&\quad + \sin\beta\cos^2\gamma\sin^2\alpha\cos\beta + \sin^2\beta\cos^2\gamma\sin\alpha\cos\alpha) \\
&= 2R^2 e
\end{aligned}$$

Since "arithmetic mean" $\leq$ "geometric mean",

$$\begin{aligned}
\frac{e}{8} &\geq (\sin^8\alpha\cos^8\alpha\sin^8\beta\cos^8\beta\sin^8\gamma\cos^8\gamma)^{\frac{1}{8}} \\
&= \sin\alpha\cos\alpha\sin\beta\cos\beta\sin\gamma\cos\gamma = \frac{1}{8}\sin 2\alpha\sin 2\beta\sin 2\gamma.
\end{aligned}$$

Therefore $(A_1B_1C_1) \geq (ABC)$ as required.

Problem 3 (solution by Y Y Yeoh): For any point p, there is a circle, centred at p, which contains at least k of the points. Call this set of points A_p. Let $B_p = \{A_q : p \in A_q\}$. Then $\sum |B_p| = \sum |A_p| \geq nk$. By the *pigeonhole principle* there is a point a such that $|B_a| \geq k$. Let $A_{p_i} \in B_a$, $i = 1, 2, \ldots, k$. In each A_{p_i}, there are at least $k-1$ points other than a. This gives a total of at least $k(k-1)$ points (counting repetitions), other than a. (Note that each p_i may be in A_{p_j} for some $j \neq i$.)

These k sets of B_a intersect in at most $\binom{k}{2}$ points other than a. So at most $\binom{k}{2}$ points are in more than 1 A_{p_i}'s. Thus $k(k-1)+1-\binom{k}{2} \leq n$, i.e., $k^2 - k + 2 - 2n \leq 0$, i.e.,

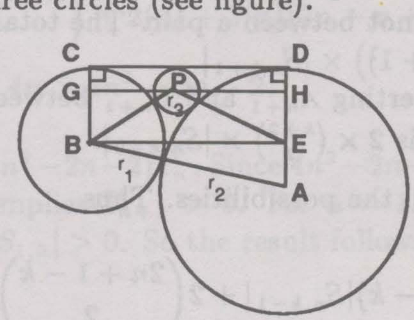
$$\frac{1}{2} - \sqrt{2n - \frac{3}{2}} \leq k \leq \frac{1}{2} + \sqrt{2n - \frac{3}{2}} < \frac{1}{2} + \sqrt{2n},$$

as required.

(Note: This solution does not rely on condition (i).)

Problem 4: Solutions by M Y Lee, L K Ng, Y Y Yeoh and C K Yu are similar to the following.

Draw a circle with centre A and radius AD and a circle with centre B and radius BC. Then these two circles are tangent to each other. If a circle centred at P with radius h is drawn, then this circle is tangent to the other two circles. The maximum value of h is attained when CD is tangent to all the three circles (see figure).



Through B and P, draw lines BE and GH, respectively, parallel to CD. Then $CD = BE = GH$. Denoting the radii of the circles by r_1, r_2 and r_3 as indicated in the figure, we have, by *Pythagoras Theorem*, $BE = \sqrt{(r_1+r_2)^2 - (r_2-r_1)^2}$, $GH = \sqrt{(r_1+r_3)^2 - (r_1-r_3)^2} + \sqrt{(r_2+r_3)^2 + (r_2-r_3)^2}$. Therefore $\sqrt{r_3} = \sqrt{r_1 r_2}/(\sqrt{r_1} + \sqrt{r_2})$. Since r_3 is the maximum value of h, the required inequality follows.

Problem 5 (solution by C K Yu): Consider the following set of n consecutive integers: $\{[(n+1)!]^2+k : k=2,3\ldots,n+1\}$. For each $k=2,3,\ldots,n+1$, k^2 divides $[(n+1)!]^2$. Therefore

$$a=[(n+1)!]^2+k=k^2m+k=k(km+1)$$

for some positive integer m. If k is not a power of a prime, then a is not a power of a prime. If k is a power of a prime p, then p does not divide $km+1$, whence a cannot be a power of a prime either.

Problem 6 (solution by M Y Lee): The problem is equivalent to the following. A permutation of the set $\{A_1,B_1,A_2,B_2,\ldots,A_n,B_n\}$ is said to have property Q if A_i is beside B_i for at least one $i\in\{1,2,\ldots,n\}$. Show that, for each n, there are more permutations with property Q than without.

Each occurrence of A_i beside B_i for some i in a permutation is called a pair. Let $S_{n,k}$ denote the set of all permutations of $\{A_1,B_1,\ldots,A_n,B_n\}$ with exactly k pairs. A member of $S_{n+1,k}$ can be obtained

(1) from $S_{n,k-1}$ by inserting the pair A_{n+1},B_{n+1} into a space not between a pair. The total number is $2\times\big(2n+1-(k-1)\big)\times|S_{n,k-1}|$.
(2) from $S_{n,k}$ by either inserting each of A_{n+1} and B_{n+1} into a space not between a pair, or inserting the pair A_{n+1},B_{n+1} into a space between a pair. The total number is $2\times\binom{2n+1-k}{2}\times|S_{n,k}|+2\times k\times|S_{n,k}|$.
(3) from $S_{n,k+1}$ by inserting one of A_{n+1},B_{n+1} between a pair and the other into a space not between a pair. The total number is $2\times(k+1)\times\big(2n+1-(k+1)\big)\times|S_{n,k+1}|$.
(4) from $S_{n,k+2}$ by inserting A_{n+1} and B_{n+1} between two distinct pairs. The total number is $2\times\binom{k+2}{2}\times|S_{n,k+2}|$.

The above cover all the possibilities. Thus

$$\begin{aligned}|S_{n+1,k}|=&2(2n+2-k)|S_{n,k-1}|+2\binom{2n+1-k}{2}|S_{n,k}|+2k|S_{n,k}|\\&+2(k+1)(2n-k)|S_{n,k+1}|+2\binom{k+2}{2}|S_{n,k+2}|.\end{aligned}$$

Consider the expression

$$T_{n+1}=|S_{n+1,n+1}|+|S_{n+1,n}|+\cdots+|S_{n+1,1}|-|S_{n+1,0}|.$$

This can be written in terms of $|S_{n,i}|$ using the above recurrence relation. The coefficient of $|S_{n,0}|$ is

$$-2\binom{2n+1}{2}+2\binom{2n+2-1}{1}=-4n^2+2n+2.$$

The coefficient of $|S_{n,1}|$ is

$$2(10)(2n)+\left[2\binom{2n}{2}+2\right]+4n=4n^2-2n+2>4n^2-2n-2.$$

The coefficient of $|S_{n,2}|$ is

$$-2(1)+2(2)(2n-1)+2\binom{2n-1}{2}+2(2)=4n^2+2n>4n^2-2n-2.$$

The coefficient of $|S_{n,i}|$, $3\le i\le n-1$, is

$$2\binom{2n+1-i}{2}+2\binom{2n+1-i}{2}+2i+2i(2n-i+1)+2\binom{i}{2}$$
$$=4n^2+6n+2>4n^2-2n-2.$$

The coefficient of $|S_{n,n}|$ is

$$2\binom{2n+1-n}{2}+2(n)+2(n)(2n-n+1)+2\binom{n}{2}$$
$$=4n^2+4n>4n^2-2n-2.$$

Therefore $T_{n+1}>(4n^2-2n-2)T_n$. Since $4n^2-2n-2=(n-1)(4n+2)\ge 0$ for $n\ge 1$, $T_n>0$ implies $T_{n+1}>0$. For $n=1$, $|S_{1,0}|=0$, $|S_{1,1}|=2$. Thus $T_1=|S_{1,1}|-|S_{1,0}|>0$. So the result follows by induction.

30th IMO(1989) in W. Germany

Participating countries and distribution of awards

Country	Team Size	Score	Medals			Honourable Mention
			Gold	Silver	Bronze	
1. People's Republic of China	6	237	4	2	-	-
2. Rumania	6	223	2	4	-	-
3. USSR	6	217	3	2	1	-
4. German Democratic Republic	6	216	3	2	1	-
5. USA	6	207	1	4	1	-
6. Czechoslovakia	6	202	2	1	3	-
7. Bulgaria	6	195	1	3	2	-
8. Federal Republic of Germany	6	187	1	3	2	-
9. Vietnam	6	183	2	1	3	-
10. Hungary	6	175	-	4	1	1
11. Yugoslavia	6	170	1	3	1	1
12. Poland	6	157	-	3	3	-
13. France	6	156	-	1	5	-
14. Iran	6	147	-	2	3	1
15. Singapore	6	143	-	-	4	2
16. Turkey	6	133	-	1	4	1
17. Hong Kong	6	127	-	2	1	1
18. Italy	6	124	-	1	2	3
19. Canada	6	123	-	1	3	2
20. Greece	6	122	-	1	3	2
21. United Kingdom	6	122	-	2	1	2
22. Australia	6	119	-	2	2	-
22. Colombia	6	119	-	1	2	3
24. Austria	6	111	-	2	1	1
25. India	6	107	-	-	4	1
26. Israel	6	105	-	2	1	-
27. Belgium	6	104	-	-	3	2
28. Republic of Korea	6	97	-	1	-	4
29. Netherlands	6	92	-	1	1	2
30. Tunisia	6	81	-	1	-	2
31. Mexico	6	79	-	-	1	3
32. Sweden	6	73	-	-	2	1
33. Cuba	6	69	-	-	1	3
33. New Zealand	6	69	-	-	2	2
35. Luxemburg	3	65	-	1	1	-
36. Brazil	6	64	-	-	3	-
36. Norway	4	64	-	-	1	2
38. Morocco	6	63	-	-	1	3
39. Spain	6	61	-	-	1	4
40. Finland	6	58	-	-	-	3
41. Thailand	6	54	-	-	1	2
42. Peru	6	51	-	-	-	3
43. Philippines	6	45	-	1	-	-
44. Portugal	6	39	-	-	-	4
45. Ireland	6	37	-	-	-	2
46. Iceland	4	33	-	-	-	2
47. Kuwait	6	31	-	-	-	-
48. Cyprus	6	24	-	-	-	1
49. Indonesia	6	21	-	-	-	-
50. Venezuela	4	6	-	-	-	-