Singapore Mathematical Society

Interschool Mathematical Competition 1989

Part A

Saturday, 17 June 1989 1000–1100

Attempt as many questions as you can.
No calculators are allowed.
Circle your answers on the Answer Sheet provided.
Each question carries 5 marks.

1. Suppose that each of 5 sticks is broken into two parts, one of which is coloured white and the other coloured red. These 10 parts are then mixed and arranged at random into 5 pairs. The probability that each white part is paired with a red part is

(A) $\frac{1}{252}$ (B) $\frac{1}{120}$ (C) $\frac{8}{63}$ (D) $\frac{4}{15}$ (E) None of the preceding

2. Let $f(x) = 1 + x(1 - x(1 + x(1 - x(1 + x \cdots))))$, for $|x| < 1$. Then $f(x) =$

(A) $\frac{1}{1+x^2}$ (B) $\frac{1}{1+x}$ (C) $\frac{1}{1-x^2} + x$ (D) $\frac{1+x}{1+x^2}$ (E) $\frac{1+x}{1+x^3}$

3. Let A be any set of 19 distinct numbers chosen from the arithmetic progression 1, 4, 7, ... , 100. Then there are always two distinct numbers in A whose sum is

(A) 92 (B) 104 (C) 110 (D) 113 (E) 116

4. A wire of length 2π is cut into three pieces. Two squares of sides x, y, respectively and a circle of radius r are constructed out of the three pieces. The sum of the areas of the squares and the circle will be a maximum when the ordered triple (x, y, r) equals

(A) $(\frac{\pi}{8}, \frac{\pi}{8}, \frac{1}{2})$ (B) $(\frac{\pi}{4}, 0, \frac{1}{2})$ (C) $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$
(D) $(0, 0, 1)$ (E) None of the preceding

5. The value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{5}}}$ is

(A) 0 (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{4}$ (D) π (E) None of the preceding

6. Let f be a function such that

$$|f(x)-f(y)| \leq |x-y|^2$$

for all real x and y. If $f(0)=1$, then the value of $\int_0^1 f(x^2)dx$ is

(A) 0 (B) 1 (C) -1 (D) 2 (E) None of the preceding

7. Suppose ΔABC is equilateral and D is a point outside it such that $\angle ADB = 90°$ and $AD = 2$. If the distance between C and the line DB is 5, then the length of DB is

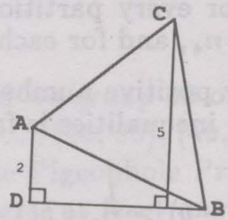
(A) 4

(B) 5

(C) $7/\sqrt{2}$

(D) $8/\sqrt{3}$

(E) $9/\sqrt{3}$



8. The figure below shows 3 circles touching in pairs. The outer circle is of radius 3. The radii of the two inner circles are 1 and 2, respectively. The radius of a circle that touches these three circles is

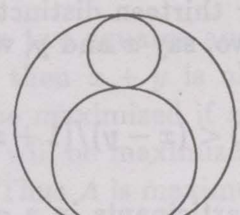
(A) 3/7

(B) 4/7

(C) 5/7

(D) 6/7

(E) 1



9. Adam, Bobby, Charles and Dave are able to correctly guess the outcome of throwing a fair coin with probabilities 0.6, 0.7, 0.8 and 0.9, respectively. Now a fair coin is thrown in a concealed box and all of them make their guesses independently. What is the probability that it is a tail, given that all except Dave guess that it is a tail ?

(A) $\frac{14}{23}$ (B) $\frac{189}{625}$ (C) $\frac{436}{625}$ (D) $\frac{604}{625}$ (E) $\frac{1}{2}$

10. Let $P(x)$ be a polynomial of degree 3. For each $n = 1, 2, 3, 4$, the remainder of $P(x)$ when divided by $x-n$ is $\frac{1}{n}$. The remainder of $P(x)$ when divided by $x-5$ is

(A) 5 (B) 1/5 (C) 1 (D) 0 (E) None of the preceding

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Part B

Saturday, 17 June 1989 1100–1300

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 25 marks.

1. Let $n \geq 5$ be an integer. Show that n is a prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 positive integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation (i, j, p, q) of $(1, 2, 3, 4)$.

2. Given arbitrary positive numbers a, b and c, prove that at least one of the following inequalities is false :

$$a(1-b) > \frac{1}{4}, \quad b(1-c) > \frac{1}{4}, \quad c(1-a) > \frac{1}{4}.$$

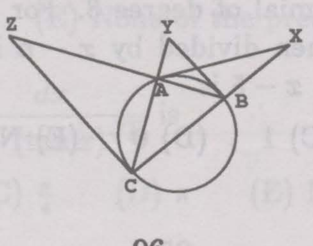
3. a. Show that $\tan \frac{\pi}{12} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$.

   b. Given any thirteen distinct real numbers, show that there exist at least two, say x and y, which satisfy the inequality

$$0 < (x-y)/(1+xy) < \sqrt{(2-\sqrt{3})/(2+\sqrt{3})}.$$

4. There are n participants in a conference. Suppose (i) every 2 participants who know each other have no common acquaintances; and (ii) every 2 participants who do not know each other have exactly 2 common acquaintances. Show that every participant is acquainted with the same number of people in the conference.

5. In the following diagram, ABC is a triangle, and X, Y and Z are respectively the points on the sides CB, CA and BA extended such that XA, YB and ZC are tangents to the circumcircle of ΔABC. Show that X, Y and Z are collinear.



Interschool Mathematical Competition 1989
Solutions
Part A

1. In general, if P_{2n} is the required probability when n sticks are broken, then $P_{2n} = \frac{n}{2n-1} P_{2n-2}$ because the probability that the first red part is paired with a white part is $\frac{n}{2n-1}$ while the probability that the remaining parts are paired as required is P_{2n-2}. Since $P_2 = 1$, $P_{10} = \frac{8}{63}$.

2. Clearly $1 + x(1 - xf(x)) = f(x)$. Thus $1 + x - x^2 f(x) = f(x)$, whence $f(x) = (1+x)/(1+x^2)$.

3. There are 34 numbers in all and we group 32 of them into 16 pairs that add up to 104: $(4, 100), (7, 97), \ldots, (46, 58), (49, 55)$. The remaining two numbers are 1 and 52. Using the Pigeonhole Principle, it is easy to see that in any selection of 19 numbers, at least one of the 16 pairs will always be present. Using the same method, it is easy to check that A, C, D, E are not possible for some suitable selections. Therefore the answer is B.

4. Let A be the total area of the two squares and the circle. Then $A = x^2 + y^2 + \pi r^2$. If r is fixed, then $x + y$ is also fixed and $A = x^2 + y^2 +$ constant. Clearly A will be maximized if $x = 0$ or $y = 0$, say $x = 0$. Then $A = y^2 + \pi r^2$. Again A will be maximized if $y = 0$ or $r = 0$, If $r = 0$, $A = \pi^2/4$. If $y = 0$, $A = \pi$. Thus A is maximum when $x = y = 0$ and $r = 1$.

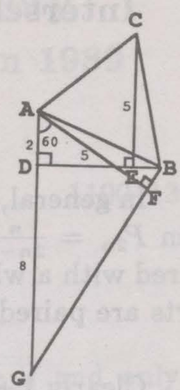
5. Let the given integral be I. Then using the substitution $y = \frac{\pi}{2} - x$, we have

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan(\frac{\pi}{2} - x))^{\sqrt{5}}} = \int_0^{\frac{\pi}{2}} \frac{(\tan x)^{\sqrt{5}}}{1 + (\tan x)^{\sqrt{5}}} dx.$$

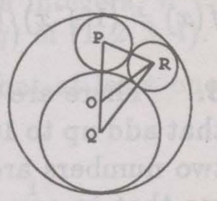
Therefore $2I = \int_0^{\frac{\pi}{2}} dx = \pi/2$, whence $I = \pi/4$.

6. We have $\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|$. By letting $x \to y$, we have $f'(x) = 0$. Thus $f(x)$ is constant for all x. Since $f(0) = 1$, we have $f(x) = 1$ for all x. So the value of the integral is 1.

7. Let E be the foot of the perpendicular from C to DB. Rotate the figure counterclockwise about B by $60°$, so that C goes to A, and E to F. Clearly $\angle AFB = 90°$. Since CE has been rotated by $60°$ to AF, $\angle DAF = 60°$. Therefore $AG = 2AF = 2CE = 10$, whence $DG = 8$. This gives $DB = 8\tan 30° = 8/\sqrt{3}$.



8. Let O be the centre of the outer circle and P, Q be the centers of the two inner circles whose radii are 1 and 2, respectively. Then O, P and Q are collinear. If R is the centre of the third inner circle with radius r, then $PR = 1 + r$, $QR = 2 + r$, $OR = 3 - r$. Since $PO = 2$, $OQ = 1$, the area of ΔOPR is twice that of ΔOQR. Now the are of a triangle with sides a, b, c is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter. The semi-perimeters of ΔOPR and ΔOQP are both 3. So we have $3(3-2)(3-1-r)(3-3+r) = 4 \times 3(3-1)(3-3+r)(3-2-r)$. Solving this equation gives $r = 6/7$.



9. If it is a tail, the probability that all except Dave will guess so is $0.6 \times 0.7 \times 0.8 \times 0.1 = 0.0336$. If it is a head, the probability that all except Dave will guess otherwise is $0.4 \times 0.3 \times 0.2 \times 0.9 = 0.0216$. Thus the required probability $= \frac{0.0336}{0.0336+0.0216} = \frac{14}{23}$.

10. By the Remainder Theorem, $P(n) = 1/n$ for each $n = 1, 2, 3, 4$. Let $f(x) = xP(x) - 1$. Then $f(x)$ is a polynomial of degree 4 and $f(n) = 0$ for each $n = 1, 2, 3, 4$. Hence, $x-1$, $x-2$, $x-3$ and $x-4$ are factors of $f(x)$. Hence $f(x) = xP(x) - 1 = k(x-1)(x-2)(x-3)(x-4)$, where k is a constant. But $f(0) = k(-1)(-2)(-3)(-4) = -1$. Hence $k = -\frac{1}{24}$. This gives $5P(5) - 1 = -1$, i.e., $P(5) = 0$. Thus the required remainder is 0.

Part B

1. If n is not a prime, we can write $n = (a+1)(b+1) = ab + a + b + 1$, where a and b are positive integers, giving a partition of n with the required property.

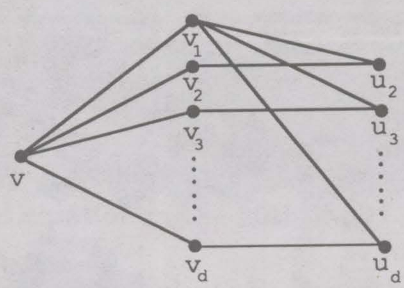
Conversely, suppose $n = n_1 + n_2 + n_3 + n_4$ and $n_1 n_2 = n_3 n_4$. Let $n_1/n_4 = n_3/n_2 = h/k$, where h and k are positive integers with no common factor. Write $n_1 = hr$, $n_2 = ks$; $n_3 = hs$, $n_4 = kr$. Then $n = hr + ks + hs + kr = (h+k)(r+s)$. So n is not a prime.

2. Observe that $a(1-b)b(1-c)c(1-a) = a(1-a)b(1-b)c(1-c)$. Now for any positive number x, we have $x(1-x) \le \left(\frac{x+1-x}{2}\right)^2 = \frac{1}{4}$. Thus $a(1-b)b(1-c)c(1-a) \le \frac{1}{64}$. Therefore at least one of $a(1-b)$, $b(1-c)$, $c(1-a)$ is $\le \frac{1}{4}$.

3. Let $\theta = \frac{\pi}{12}$. Then $\tan 2\theta = 1/\sqrt{3}$. But $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta) = 1/\sqrt{3}$. Solving for $\tan\theta$ yields the result for part (a).

For part (b), let the given numbers be $a_1, a_2, \ldots, a_{13}$. Since $\tan x$ maps $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $(-\infty, \infty)$, we can find distinct $u_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan u_i = a_i$, $1 \le i \le 13$. Partition $(-\frac{\pi}{2}, \frac{\pi}{2})$ into 12 equal parts. By the Pigeonhole Principle, at least one of the subintervals must contain two u_i's, say u_j and u_k. Let $x = \tan u_j$, $y = \tan u_k$, $x > y$. Then $0 < \tan(u_j - u_k) < \tan\frac{\pi}{12}$, and we get $0 < \frac{x-y}{1+xy} < \tan\frac{\pi}{12} = \sqrt{(2-\sqrt{3})/(2+\sqrt{3})}$.

4. Let G be the graph whose vertices are the participants, and two vertices are adjacent if and only if the corresponding participants know each other. Let v be a vertex with maximum degree d and $v_1, v_2, \ldots, v_d$ be the vertices adjacent to v. Consider any vertex u which is not adjacent to v. By (ii), u and v are adjacent to two common vertices, one of which is one of the v_i's. Thus G is connected.



By condition (i) v_i and v_j are not adjacent if $1 \le i < j \le d$. Consider v_1. Since the maximum degree is d, the degree of v_1 is at most d. Since v_1 and v_2 are not adjacent, by condition (ii), there is exactly one vertex, say u_2, distinct from v which is adjacent to v_1 and v_2. Similarly, for each $j = 3, 4, \ldots, d$, there exists a vertex u_j distinct from v and adjacent to v_1 and v_j. By (ii) again, these u_i's are distinct. For if $u_i = u_j$, then v and u_i will be adjacent to v_1, v_i and v_j, contradicting (ii). Thus the degree of v_1 is at least d. But the degree of v_1 is at most d. Therefore v_1 is adjacent to exactly d vertices. Since G is connected, the preceding argument shows that every vertex of G has degree d as required.

5. By applying Menelaus' Theorem to ΔABC, it suffices to prove that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

For any ΔPQR let (PQR) denote its area. Observe that $(AXB)/(AXC) = XB/XC$. Also since $\angle XAB = \angle ACX$ we have $\Delta AXB \simeq \Delta CXA$, whence $(AXB)/(CXA) = (AB/AC)^2$. Therefore $BX/XC = (AB/AC)^2$. Similarly, $CY/YA = (BC/AB)^2$ and $AZ/ZB = (AC/BC)^2$. Thus

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \left(\frac{AB}{BC} \cdot \frac{BC}{AB} \cdot \frac{AC}{BC}\right)^2 = 1$$

as required.

From left: Lin Ziwei, Cai Zheming, Huang Zhiwei(from Chinese High School, winning team of the Inter-Secondary School Mathematical Competition.) Professor Louis H Y Chen, President of Singapore mathematical Society. Professor Sir Michael Atiyah who gave away the prizes. Professor Peng Tsu Ann, Head, Department of Mathematics, National University of Singapore. Yeoh Yong Yeow, Lam Vui Chap, Lee Mun Yew (from Raffles Junior College, winning team of the Interschool Mathematical Competition.)