Singapore Mathematical Society Interschool Mathematical Competition 1988

A total of 300 students from 12 junior colleges and 41 secondary schools took part in the Interschool Mathematical Competition on Saturday, 25 June 1988, at the National University of Singapore. The winners of the team and individual awards are as follows:

Team Results		
Position	School	Team Members
1	Hwa Chong Junior College	Ngan Ngiap Teng Ng Kien Ming Lim Jing Yee
2	National Junior College	Chan Ying Yin Cheong Kok Wui Loh Ngai Seng
3	Raffles Junior College	Ng Lup Keen Lee Mun Yew Yeoh Yong Yeow
4	Victoria Junior College	Low Teng Yong Tan Kee Sian Soo Sai Shyong

Individual Results		
Position	Competitor	School
1	Cheong Kok Wui	National Junior College
1	Yeoh Yong Yeow	Raffles Junior College
3	Ngan Ngiap Teng	Hwa Chong Junior College
3	Ng Kien Ming	Hwa Chong Junior College
5	Lim Jing Yee	Hwa Chong Junior College

The Singapore Mathematical Society Challenge Shield was won by *Hwa Chong Junior College. Cheong Kok Wui* and *Yeoh Yong Yeow* were jointly awarded the Southeast Asian Mathematical Society Prize. *Yan Weide* of The Chinese High School, *Yu Changkai* of Dunman High School and *Xie Zhiqun* of Raffles Institution were awarded a prize each for being the best competitors from the secondary schools.

The prizes were presented by Professor Edwin Hewitt, Professor Emeritus of University of Washington on 30 August 1988. He also gave a lecture entitled *The Riddle of Primes* (see this issue). The problems and solutions for this year's competition are printed below.

Problems

Part A

Saturday, 25 June 1988 1000-1100

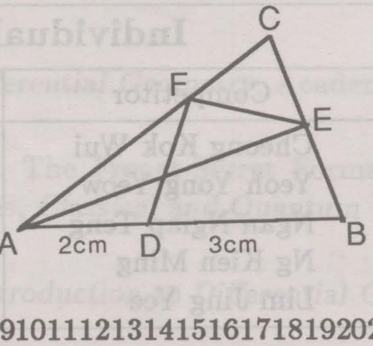
Attempt as many questions as you can.
No calculators are allowed.
Circle your answers on the Answer Sheet provided.
Each question carries 5 marks.

1. Suppose $X_1, X_2, \ldots, X_{10}$ are any positive integers such that $X_1+X_2+\cdots+X_{10}=50$. Then the maximum value of $X_1^2+X_2^2+\cdots+X_{10}^2$ is

   (a) 1988 (b) 1780 (c) 1690 (d) 1560 (e) none of the above.

2. Suppose area of $\Delta ABC = 10$ cm^2, $AD = 2$ cm, $DB = 3$ cm and area of ΔABE = area of quadrilateral $DBEF$. Then area of ΔABE equals

   (a) 4 cm^2
   (b) 5 cm^2
   (c) 6 cm^2
   (d) $\dfrac{5}{3\sqrt{10}}$ cm^2
   (e) $\dfrac{5}{2\sqrt{10}}$ cm^2.



3. Consider the number $0.1234567891011121314151617181920 21\cdots$. The digit at the 1988^{th} decimal place is

   (a) 0 (b) 3 (c) 4 (d) 8 (e) 9.

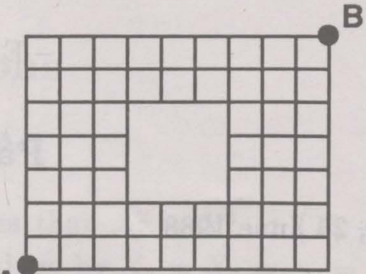
4. The value of $\dfrac{1}{2\sqrt{1}+1\sqrt{2}}+\dfrac{1}{3\sqrt{2}+2\sqrt{3}}+\cdots+\dfrac{1}{100\sqrt{99}+99\sqrt{100}}$ is

(a) $\frac{3}{4}$ (b) $\frac{9}{10}$ (c) 1 (d) $\sqrt{2}$ (e) none of the above.

5. The maximum value of the function $y=\sqrt{1+sinx}+\sqrt{1-sinx}$ is

(a) 2 (b) $\sqrt{5}$ (c) $\sqrt{6}$ (d) $2\sqrt{2}$ (e) none of the above.

6. How many shortest paths are there for an ant to crawl from A to B along the grid as shown below?



(a) 3560
(b) 3650
(c) 3600
(d) 3500
(e) none of the above.

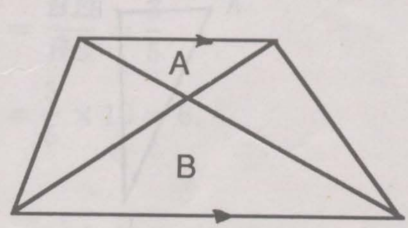
7. How many true statements are there among the following four statements?

(i) $1=2 \Rightarrow 10^2=100$ (ii) $1=2 \Rightarrow 10^2 \neq 100$
(iii) $1 \neq 2 \Rightarrow 10^2=100$ (iv) $1 \neq 2 \Rightarrow 10^2 \neq 100$

(a) None (b) 1 (c) 2 (d) 3 (e) 4

8. A trapezium is divided into 4 triangles by its diagonals as shown below. Let A and B denote the areas of the triangles adjacent to the parallel sides. The area of the trapezium is

(a) $2(A+B)$
(b) $(\sqrt{A}+\sqrt{B})^2$
(c) $A+B+\sqrt{AB}$
(d) $A+B+\dfrac{AB}{A+B}$
(e) none of the above.



9. A store contains 10 T.V. sets, 3 of which are known to be faulty. The sets are examined one at a time in a random order. The probability

that all the faulty sets will have been discovered within the inspection of the first 5 sets is

(a) $\frac{3}{10}$ (b) $\frac{1}{12}$ (c) $\frac{2}{7}$ (d) $\frac{1}{7}$ (e) none of the above.

10. Let C be a circle centred at $(\sqrt{2}, \sqrt{3})$ with radius r, where r is any positive real number. Call a point (x, y) *rational* if both x and y are rational numbers. Then the greatest possible number of rational points that lie on C is

(a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many.

– E N D –

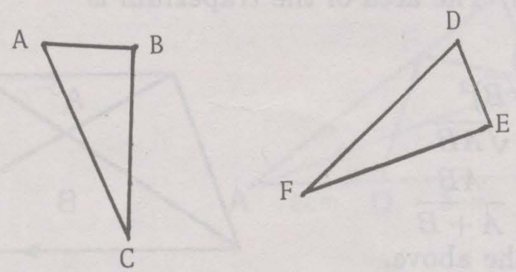
Part B

Saturday, 25 June 1988 1100-1300

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 25 marks.

1. Let $f(x)$ be a polynomial of degree n such that $f(k) = \dfrac{k}{k+1}$ for each $k = 0, 1, 2, \ldots, n$. Find $f(n+1)$.

2. Suppose ΔABC and ΔDEF in the figure below are congruent. Prove that the perpendicular bisectors of AD, BE and CF intersect at the same point.



3. Find all the positive integers n such that P_n is divisible by 5, where $P_n = 1 + 2^n + 3^n + 4^n$. Justify your answer.

4. Prove that for any positive integer n, any set of $n+1$ distinct integers chosen from the integers $1, 2, \ldots, 2n$ always contains 2 distinct integers such that one of them is a multiple of the other.

5. Find all positive integers x, y, z satisfying the equation

$$5(xy + yz + zx) = 4xyz.$$

– E N D –

Solutions

Part A

1. Observe that $1 < X_i < X_j$ implies that $X_i^2 + X_j^2 < (X_i - 1)^2 + (X_j + 1)^2$. So the maximum is attained by $X_1 = X_2 = \cdots = X_9 = 1$ and $X_{10} = 41$. The answer is therefore c.

2. Observe that

$$\text{area}(ABE) = \text{area}(ADE) + \text{area}(DBE)$$

and $$\text{area}(DBEF) = \text{area}(DEF) + \text{area}(DBE).$$

Therefore $$\text{area}(ADE) = \text{area}(DEF).$$

Hence distance from F to DE = distance from A to DE. Therefore DE is parallel to AC. By similar triangles, $BE/BC = 3/5$. Since

$$\frac{\text{area}(ABE)}{\text{area}(ABC)} = \frac{BE}{BC} = \frac{3}{5},$$

we have $$\text{area}(ABE) = \frac{3}{5} \times 10 = 6.$$

3. Observe that there are 9 one-digit numbers (1–9), 90 two-digit numbers (10–99) and 900 three-digit numbers (100–999). Also

$$1988 = 9 + 90 \times 2 + 599 \times 3 + 2.$$

So we have
$$0.\underbrace{1\cdots 9}_{\substack{9\\ 1-digit}}\ \underbrace{10\cdots 99}_{\substack{90\\ 2-digit}}\ \underbrace{100\cdots 698}_{\substack{599\\ 3-digit}}\ 69\underline{9},$$

and the 1988^{th} decimal place is 9.

4. The sum is

$$\sum_{n=1}^{99} \frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} = \sum_{n=1}^{99} \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)n}$$
$$= \sum_{n=1}^{99}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$
$$= 1 - \frac{1}{\sqrt{100}} = \frac{9}{10}.$$

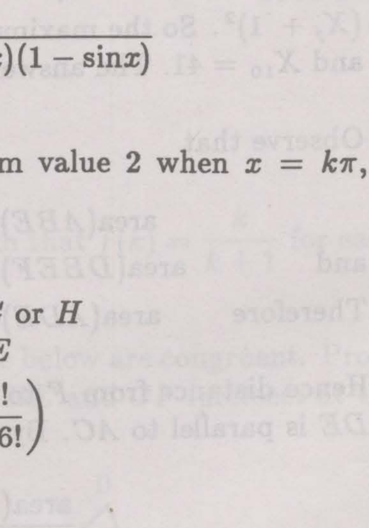
5. Since
$$y^2 = 2 + 2\sqrt{(1+\sin x)(1-\sin x)}$$
$$= 2 + 2|\cos x|$$

and $|\cos x| \le 1$, y attains its maximum value 2 when $x = k\pi$, k integer.

6. The required no. of ways
= Total no. of ways via C, D, E, F, G or H
= $2 \times$ Total no. of ways via C, D, or E
$$= 2\left(\frac{8!}{7!1!}\times 1 + \frac{8!}{6!2!}\times\frac{8!}{1!7!} + \frac{8!}{5!3!}\times\frac{8!}{2!6!}\right)$$
$=3600$.

Second Method

The required no. of ways
= Total no. of ways from A to B
− Total no. of ways via P or Q

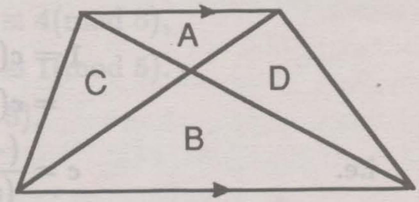


$$= \frac{16!}{9!7!} - \left(\frac{8!}{4!4!}\times\frac{8!}{5!3!} + \frac{8!}{5!3!}\times\frac{8!}{4!4!}\right)$$
$$= \frac{16!}{9!7!} - 2\times\frac{8!}{4!4!}\times\frac{8!}{5!3!}$$
$= 3600$.

7. A statement of the form $p \Longrightarrow q$ is false if and only if p is true and q is false. Thus the only false statement is (iv). Hence the answer is d.

8. Let C and D denote the area of the other two triangles (see figure). Observe that $C = D$. Thus

$$\frac{C}{B} = \frac{x}{y} = \frac{A}{D} = \frac{A}{C}.$$



Therefore

$$\text{area} = A + B + 2\sqrt{AB} = (\sqrt{A} + \sqrt{B})^2.$$

9. There are $\binom{10}{3}$ ways to arrange the sets in a linear order. There are $\binom{5}{3}$ arrangements in which the faulty sets occur among the first 5 sets. Thus the required probablity is $\binom{5}{3}/\binom{10}{3} = 1/12$.

10. Suppose (x_1, y_1), (x_2, y_2) are two distinct rational points on the circle. Then

$$x_1^2 - 2\sqrt{2}x_1 + 2 + y_1^2 - 2\sqrt{3}y_1 + 3$$
$$= x_2^2 - 2\sqrt{2}x_2 + 2 + y_2^2 - 2\sqrt{3}y_2 + 3,$$

i.e. $$(x_1^2 - x_2^2) + (y_1^2 - y_2^2) = 2\sqrt{2}(x_1 - x_2) + 2\sqrt{3}(y_1 - y_2).$$

In the latter equation the left side is a rational number and the right side is an irrational number since at least one of $x_1 - x_2$ and $y_1 - y_2$ is not 0. This is impossible. So the answer is b.

Part B

1. Since $(k+1)f(k) - k = 0$, the $(n+1)^{\text{th}}$ degree polynomial

$$g(x) = (x+1)f(x) - x$$

will have $x = 0, 1, \ldots, n$ as its $(n+1)$ roots. Hence

$$g(x) = cx(x-1)\cdots(x-n)$$

where c is a constant,

i.e. $$(x+1)f(x) - x = cx(x-1)\cdots(x-n).$$

To determine c, we put $x = -1$ to get

$$\begin{aligned} 1 &= c(-1)(-2)\cdots(-1-n) \\ &= c(-1)^{n+1}(n+1)!, \end{aligned}$$

i.e. $$c = \frac{(-1)^{n+1}}{(n+1)!}.$$

Hence when $x = n+1$, we have

$$(n+2)f(n+1) - (n+1) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)n\cdots 1$$

i.e. $$f(n+1) = \frac{(-1)^{n+1} + n + 1}{n+2}.$$

2. Let O be the intersection point of the perpendicular bisectors of CF and BE. Then

$$OC = OF,\ OB = OE,\ BC = EF.$$

Thus $$\Delta BCO \equiv \Delta EFO$$

and $$\angle ACO = \angle DFO.$$

Also $AC = DF$, $OC = OF$. Therefore

$$\Delta ACO \equiv \Delta DFO.$$

and $$AO = DO.$$

So the line joining O to the midpoint of AD is the perpendicular bisector of AD.

Second Method

Since ΔABC and ΔDEF are similarly oriented, the former can be transformed to the latter by a rotation with a center which we called O. Clearly $OA = OD$. So the perpendicular bisector of AD passes through O. Similarly the perpendicular bisectors of BE and CF also pass through O, and the result follows.

3. Note that

$$2 \equiv 2 (\mathrm{mod}\ 5), \qquad 2^2 \equiv 4 (\mathrm{mod}\ 5),$$
$$2^3 \equiv 3 (\mathrm{mod}\ 5), \qquad 2^4 \equiv 1 (\mathrm{mod}\ 5).$$

Therefore $\qquad 2^n \equiv 2^{n+4} (\mathrm{mod}\ 5).$

Similarly

$$3^n \equiv 3^{n+4} (\mathrm{mod}\ 5) \quad \text{and} \quad 4^n \equiv 4^{n+4} (\mathrm{mod}\ 5).$$

So $P_n \equiv P_{n+4} (\mathrm{mod}\ 5)$. Therefore it suffices to check $n = 1, 2, 3, 4$. Now

$$P_1 \equiv 0 (\mathrm{mod}\ 5), \qquad P_2 \equiv 0 (\mathrm{mod}\ 5),$$
$$P_3 \equiv 0 (\mathrm{mod}\ 5), \qquad P_4 \equiv 4 (\mathrm{mod}\ 5).$$

Hence the answer is any positive integer not divisible by 4.

4. Let $A = \{a_1, a_2, \ldots, a_{n+1}\} \subset \{1, 2, \ldots, 2n\}$. For each $i = 1, 2, \ldots, n+1$, we have

$$a_i = 2^{k_i} b_i,$$

where k_i is a non-negative integer and b_i is an odd integer such that $1 \leq b_i < 2n$. Since $\{1, 2, \ldots, 2n\}$ contains exactly n odd integers, namely 1, 3, ..., $2n$-1, we conclude that the odd integers $b_1, b_2, \ldots, b_{n+1}$ are not all distinct. Thus there exist two distinct integers $j, r \in \{1, 2, \ldots, n+1\}$ such that $b_j = b_r$. Hence a_j is a multiple of a_r if $k_j > k_r$, and a_r is a multiple of a_j if $k_r > k_j$.

5. Rewrite the equation as

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}.$$

We may assume $x \leq y \leq z$. Then

$$\frac{3}{x} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}.$$

So $x < 4$. Also

$$\frac{1}{x} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{5}.$$

So $x > 1$.

Case 1 $x = 2$:

Then

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{4}{5} - \frac{1}{2} = \frac{3}{10} \quad \text{and} \quad \frac{1}{y} < \frac{1}{y} + \frac{1}{z} = \frac{3}{10}.$$

So $y < 7$ and $y > 3$. If $y = 4$, then $z = 20$. If $y = 5$, then $z = 10$. If $y = 6$, then z is not an integer.

Case 2 $x = 3$:

As in Case 1 we have

$$\frac{2}{y} \geq \frac{4}{5} - \frac{1}{3} = \frac{7}{15} \quad \text{and} \quad \frac{1}{y} < \frac{7}{15}.$$

So $y < 5$ and $y > 2$. For $y = 3$ or $y = 4$, there is no integer solution for z.

Therefore there are 12 ordered triples (x, y, z) of positive integers satisfying the given equation: (2,4,20), (2,20,4), (4,2,20), (20,2,4), (4,20,2), (20,4,2), (2,5,10), (2,10,5), (5,2,10), (10,2,5), (5,10,2) and (10,5,2).