Curves in Four-Dimensional Space*

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§1 Introduction to Curves

1.1 Definition of a Curve

A curve in a space of any dimension can be seen as the trip taken by a moving particle. Its position at a particular instant can be determined parametrically by expressing the coordinates of the point as functions of time. Therefore, the position, β, of the particle in three dimensions is given as follows:

$$\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t)).$$

Time, t, is defined on an open interval on the real line $\mathbf{R}$. The functions β_1, β_2, and β_3 must be differentiable so that the curve will have no *corners* like A, B, C as in Figure 1.1. This also means that the particle must be moving at all times; for otherwise, situations as in Figure 1.1 may arise.

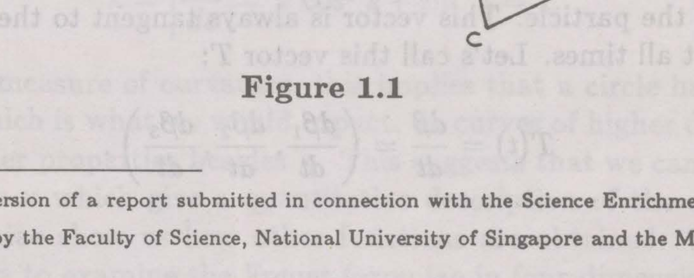

Figure 1.1

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Some examples of curves in three dimensions are :-

(1) Straight line : $\beta(t) = (p_1 + q_1 t,\ p_2 + q_2 t,\ p_3 + q_3 t)$ where $q_i \neq 0$ and $p_1, p_2, p_3, q_1, q_2, q_3$ are constants.

(2) Circle : $\beta(t) = (a\cos t, a\sin t, 0)$. This is a circle on the x-y plane with radius a and centre $(0,0,0)$.

(3) Circular helix : $\beta(t) = (a\cos t, a\sin t, bt)$. (See Figure 1.2.)

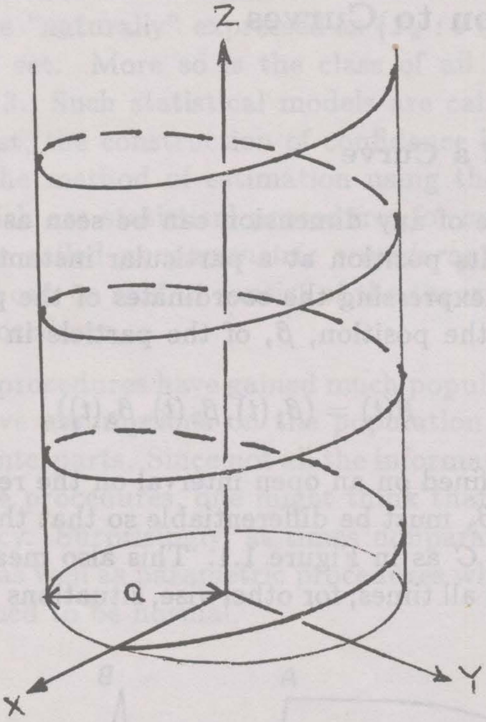

Figure 1.2

At each particular instant, there is a velocity vector representing the velocity of the particle. This vector is always tangent to the curve and is non-zero at all times. Let's call this vector T:

$$T(t) = \frac{d\beta}{dt} = \left(\frac{d\beta_1}{dt}, \frac{d\beta_2}{dt}, \frac{d\beta_3}{dt}\right).$$

1.2 Description of a Curve

In examining a curve, what is important is the shape and not the speed of the particle moving along the curve. Therefore, given any curve we can force the particle to move at a constant speed of one unit per second, i.e., $|T| = 1$. By reparametrizing its coordinates, we turn β from a function of time, t, into a function of distance along the curve, s. dT/ds now measures the rate of change of direction of T but not its magnitude.

In Figure 1.3, at point A, $|dT/ds|$ is small but at B, $|dT/ds|$ is large. Therefore, $|dT/ds|$ should be a good measure of curvature. Kappa, κ, is defined as

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{d^2\beta}{ds^2}\right|.$$

So κ is a function of s.

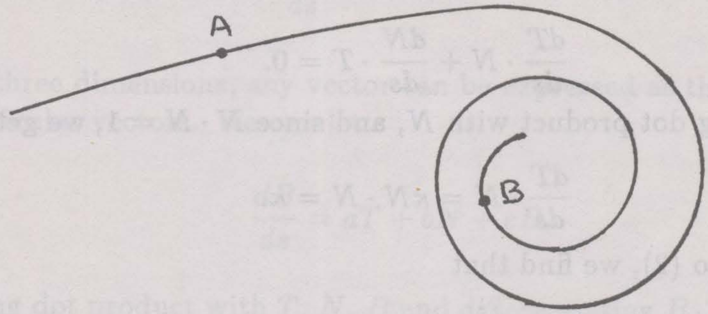

Figure 1.3

For the circle $\beta(s) = (\cos s, \sin s, 0)$,

$$\kappa = \left|\frac{dT}{ds}\right| = \sqrt{\cos^2 s + \sin^2 s} = 1.$$

Since κ is a measure of curvature, this implies that a circle has constant curvature which is what we would expect. In curves of higher dimensions, there are other properties besides κ. This suggests that we can find other functions like κ which give a quantitative description of the curve. The Frenet formulae show us how other functions are obtained. The aim of this project is to examine the Frenet formulae in four dimensions.

§2 The Frenet Formulae

2.1 The Frenet Formulae in Two Dimensions

A two-dimensional curve (i.e., contained in a plane) has curvature κ at a particular position. If T is the tangent of the curve at that point, then $T \cdot T = 1$ since the magnitude of T is one. Differentiating with respect to s, we have

$$2\frac{dT}{ds} \cdot T = 0.$$

This implies dT/ds is perpendicular to T unless dT/ds is zero. Let κ be the magnitude of dT/ds, and N its unit vector. Then

$$\frac{dT}{ds} = \kappa N. \tag{1}$$

We have $T \cdot N = 0$ since N is perpendicular to T. Differentiating, we obtain

$$\frac{dT}{ds} \cdot N + \frac{dN}{ds} \cdot T = 0. \tag{2}$$

From (1), taking dot product with N, and since $N \cdot N = 1$, we get

$$\frac{dT}{ds} \cdot N = \kappa N \cdot N = \kappa.$$

Substituting into (2), we find that

$$\frac{dN}{ds} \cdot T = -\kappa.$$

Similarly, $N \cdot N = 1$ implies

$$\frac{dN}{ds} \cdot N = 0.$$

Now any vector x can be expressed as $(x \cdot y)y + (x \cdot z)z$ where y and z are perpendicular unit vectors. Therefore,

$$\frac{dN}{ds} = \left(\frac{dN}{ds} \cdot T\right) T + \left(\frac{dN}{ds} \cdot N\right) N = -\kappa T.$$

So we get

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T,$$

which are the Frenet Formulae in two dimensions.

2.2 The Frenet Formulae in Three Dimensions

We now derive the Frenet formulae in three dimensions using a method which can be generalized to higher dimensions.

As before N is the unit vector of dT/ds and κ is the magnitude of dT/ds. Now we define a unit vector B which is perpendicular to T and N. By taking dot product with T and N we see that the vector

$$\frac{dN}{ds} - \left(\frac{dN}{ds} \cdot T\right) T - \left(\frac{dN}{ds} \cdot N\right) N$$

is perpendicular to T and N. As a result, we can let this vector be τB where τ is a scalar known as the *torsion*. By the usual method, we find that $T \cdot dN/ds = -\kappa$, $N \cdot dN/ds = 0$. Thus,

$$\frac{dN}{ds} = -\kappa T + \tau B.$$

In three dimensions, any vector can be expressed as the sum of three perpendicular vectors. We can let

$$\frac{dB}{ds} = aT + bN + cB.$$

By taking dot product with T, N, B and differentiating $B \cdot T = 0$, $B \cdot W = 0$, $B \cdot B = 0$, we find that $a = 0$, $b = -\tau$, $c = 0$. Thus the Frenet formulae are

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N.$$

2.3 The Frenet Formulae in Four Dimensions

To extend into four dimensions, we follow the same procedure till we reach $dN/ds = -\kappa T + \tau B$. By the same principle, we let D be a unit vector perpendicular to T, N and B. Then for some scalar σ,

$$\sigma D = \frac{dB}{ds} - \left(T \cdot \frac{dB}{ds}\right) T - \left(N \cdot \frac{dB}{ds}\right) N - \left(B \cdot \frac{dB}{ds}\right) B.$$

Differentiating $T \cdot B = 0$, $N \cdot B = 0$, $B \cdot B = 1$, we get

$$\frac{dT}{ds} \cdot B + \frac{dB}{ds} \cdot T = (\kappa N) \cdot B + \frac{dB}{ds} \cdot T = 0, \quad \text{i.e.,} \quad \frac{dB}{ds} \cdot T = 0;$$

$$\frac{dN}{ds} \cdot B + \frac{dB}{ds} \cdot N = (-\kappa T + \tau B) \cdot B + \frac{dB}{ds} \cdot N = 0, \quad \text{i.e.,} \quad \frac{dB}{ds} \cdot N = -\tau;$$

and $$2\frac{dB}{ds} \cdot B = 0.$$

Therefore

$$\frac{dB}{ds} = -\tau N + \sigma D.$$

Let

$$\frac{dD}{ds} = aT + bN + cB + dD.$$

By taking dot product of both sides with T, N, B and D and differentiating $T \cdot D = 0$, $N \cdot D = 0$, $B \cdot D = 0$, $D \cdot D = 1$, we get $a = b = d = 0$, $c = -\sigma$. Thus the Frenet formulae in four dimensions are:

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N + \sigma D, \quad \frac{dD}{ds} = -\sigma B.$$

From the above, we can see that the Frenet formulae for higher dimensions are likely to have the same general pattern. In fact, it can be proved by the same procedure that in each dimension, the same pattern will occur.

§3 Properties of κ, τ and σ

3.1 Properties of σ

In O'Neill's "Elementary Differential Geometry," it is shown that a three-dimensional curve with $\kappa > 0$ is a plane curve if and only if $\tau = 0$. This theorem suggests that if a curve in four dimensions has $\kappa > 0$, $\tau \neq 0$, then the curve is confined to three-dimensional space if and only if $\sigma = 0$.

Proof Let β be a curve confined to three dimensions. Then, there exist vectors x and y such that $(\beta(s) - x) \cdot y = 0$ for all s. Differentiation yields $\beta'(s) \cdot y = 0$ or $T \cdot y = 0$. Differentiating again, we get $dT/ds \cdot y = 0$. So by the Frenet formulae $(\kappa N) \cdot y = 0$, or $N \cdot y = 0$. Differentiating once more, we have $dN/ds \cdot y = 0$. By the Frenet formulae, $(-\kappa T + \tau B) \cdot y = 0$. But $T \cdot y = 0$, so $B \cdot y = 0$.

$T \cdot y = 0, N \cdot y = 0, B \cdot y = 0$ imply that y is perpendicular to T, N and B. Hence D can be written as $\pm \frac{y}{|y|}$. Differentiating, we get $dD/ds = 0$. $dD/ds = \sigma B$ by the Frenet formulae. Therefore, $\sigma B = 0$. Now $B \neq 0$, so this implies that $\sigma = 0$.

Conversely, if $\sigma = 0$, then $dD/ds = -\sigma B = 0$. This means that D must be a constant. Define a function f such that $f(s) = (\beta(s) - \beta(0)) \cdot D$ for all s. Then

$$\frac{df}{ds} = \beta'(s) \cdot D = T \cdot D = 0.$$

Thus, $f(s)$ has the same value for all s. Taking $s = 0$,

$$f(0) = (\beta(0) - \beta(0)) \cdot D = 0.$$

Therefore, $(\beta(s) - \beta(0)) \cdot D = 0$ for all s. From here, we conclude that the curve β must lie entirely in the three-dimensional space orthogonal to the vector D.

3.2 The Cylindrical Helix

The cylindrical helix is defined as a curve for which there exists a fixed unit vector u such that $T \cdot u$ is constant along the curve. By the definition of dot product,

$$T \cdot u = |T||u| \cos\theta = \cos\theta$$

where θ is the angle between T and u. Thus θ is a constant.

In O'Neill's "Elementary Differential Geometry," it is shown that τ/κ is constant along a three-dimensional helix. We will now try to find the corresponding properties of a four-dimensional helix. We have $T \cdot u = \cos\theta$. Since θ is a constant,

$$\frac{dT}{ds} \cdot u + \frac{du}{ds} \cdot T = 0.$$

Now $du/ds = 0$, so $dT/ds \cdot u = 0$. From the Frenet formulae we get $\kappa N \cdot u = 0$. Assuming that $\kappa \neq 0$, we have $N \cdot u = 0$, and

$$\begin{aligned} u &= (u \cdot T)T + (u \cdot N)N + (u \cdot B)B + (u \cdot D)D \\ &= (\cos\theta)T + \alpha B + \gamma D \end{aligned}$$

where $\alpha = u \cdot B$ and $\gamma = u \cdot D$. Thus

$$1 = |u| = \sqrt{\cos^2\theta + \alpha^2 + \gamma^2},$$

i.e.,

$$\alpha^2 + \gamma^2 = \sin^2\theta.$$

This suggests that we should define an angle ϕ by $\alpha = \sin\theta\cos\phi$, $\gamma = \sin\theta\sin\phi$ such that

$$\alpha^2 + \gamma^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = \sin^2\theta.$$

We then obtain

$$u = (\cos\theta)T + (\sin\theta\cos\phi)B + (\sin\theta\sin\phi)D.$$

Since u is a constant, differentiating and using the Frenet formulae again, we find

$$\begin{aligned} 0 &= (\cos\theta)\kappa N - \frac{d\phi}{ds}(\sin\phi\sin\theta)B + (\sin\theta\cos\phi)(-\tau N + \sigma D) \\ &\quad + \frac{d\phi}{ds}(\sin\theta\cos\phi)D + (\sin\theta\sin\phi)(-\sigma B) \\ &= (\kappa\cos\theta - \tau\sin\theta\cos\phi)N - \sin\theta\sin\phi\left(\frac{d\phi}{ds} + \sigma\right)B \\ &\quad + \sin\theta\cos\phi\left(\frac{d\phi}{ds} + \sigma\right)D. \end{aligned}$$

When a sum of non-parallel vectors is zero, the individual components are zero. Therefore,

$$\kappa\cos\theta - \tau\sin\theta\cos\phi = 0 \tag{3}$$

$$\sin\theta\sin\phi\left(\frac{d\phi}{ds} + \sigma\right) = 0 \tag{4}$$

$$\sin\theta\cos\phi\left(\frac{d\phi}{ds} + \sigma\right) = 0 \tag{5}$$

But $\sin\theta$ is a constant. If $\sin\theta = 0$, then $\theta = 0$ or π. So $T = \pm u$. Thus T is a straight line in the direction of u. This case is not of interest. So from (4) we have

$$\sin\phi = 0 \quad \text{or} \quad \frac{d\phi}{ds} + \sigma = 0.$$

By a similar reasoning, from equation (5) we get

$$\cos\phi = 0 \quad \text{or} \quad \frac{d\phi}{ds} + \sigma = 0.$$

But $\sin\phi$ and $\cos\phi$ cannot be equal to zero at the same time, so $d\phi/ds + \sigma$ must be zero. This yields $d\phi/ds = -\sigma$. Since we are interested in four-dimensional curves, so $\sigma \neq 0$, which means that ϕ is not a constant. From (3),

$$\frac{\tau}{\kappa} = \frac{\cos\theta}{\sin\theta\cos\phi} = \cot\theta\sec\phi.$$

Now $\cot\theta$ is a constant but not $\sec\phi$. Thus τ/κ is not constant.

The significance of the above discussion is that the result obtained is different from the situation in the three-dimensional case where a cylindrical helix always has τ/κ as a constant.

In the previous sections, we saw that the four-dimensional Frenet formulae and their properties resemble those of the three-dimensional formulae. The above result shows, however, that we cannot always guess the behaviour of the four-dimensional case from the three dimensional case.

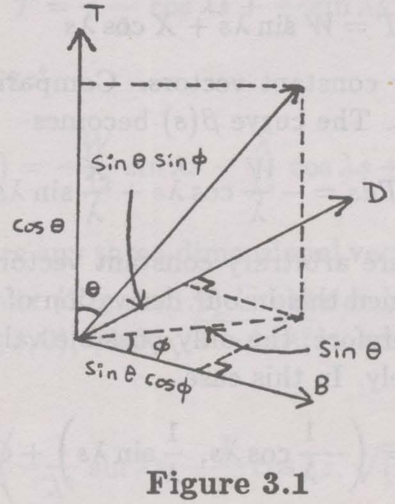

Figure 3.1

By the way, the angle ϕ does have some geometric significance. Since u is fixed, as we move along the curve, the vectors T, N, B and D will move about u in a certain way. If we neglect the vector N and fix the frame T, B and D, we will get Figure 3.1. From the figure, we see that ϕ represents the angle as shown.

§4 Simplest Curves

4.1 Simplest Curves in Two Dimensions

Apart from the straight line which is only one-dimensional, the circle is one of the simplest two-dimensional curves. The circle also has κ as constant. Therefore, we can define a *simplest curve* in two dimensions to be one in which κ is constant.

We will now investigate whether there are other two dimensional simplest curves by using the Frenet formulae :

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T.$$

Since κ is constant, differentiating the first expression, we obtain

$$\frac{d^2T}{ds^2} = \kappa \frac{dN}{ds} = -\kappa^2 T.$$

By the theory of differential equations,

$$T = W \sin \lambda s + X \cos \lambda s$$

where W and X are any constant vectors. Comparing with $d^2T/ds^2 = -\kappa^2 T$, we see that $\lambda = \kappa$. The curve $\beta(s)$ becomes

$$\beta(s) = \int T ds = -\frac{W}{\lambda} \cos \lambda s + \frac{X}{\lambda} \sin \lambda s + C.$$

Although W, X and C are arbitrary constant vectors, they must satisfy $|T| = 1$ because we assumed this in our derivation of the Frenet formulae (refer to Section 2). Therefore, the only possible value for W and X are (1,0) and (0,1) respectively. In this case

$$\beta(s) = \left(-\frac{1}{\lambda} \cos \lambda s, \ \frac{1}{\lambda} \sin \lambda s \right) + C$$

which clearly represents a circle of radius $1/\lambda = 1/\kappa$ with C as centre.

4.2 Simplest curves in Three Dimensions

A simplest curve in three dimensions has κ and τ as non-zero constants. The Frenet formulae in three dimensions are

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N.$$

Differentiating the first expression, we have

$$\frac{d^2T}{ds^2} = \kappa \frac{dN}{ds} = -\kappa^2 T + \kappa\tau B.$$

Differentiating once more,

$$\begin{aligned}\frac{d^3T}{ds^3} &= -\kappa^2 \frac{dT}{ds} + \kappa\tau \frac{dB}{ds} = -\kappa^2 \frac{dT}{ds} + \kappa\tau(-\tau N) \\ &= -\kappa^2 \frac{dT}{ds} - \tau^2 \frac{dT}{ds} = -(\kappa^2 + \tau^2)\frac{dT}{ds}.\end{aligned}$$

This is similar to the two-dimensional case. If we let $\lambda^2 = \kappa^2 + \tau^2$, then

$$\frac{dT}{ds} = W \sin \lambda s + X \cos \lambda s.$$

So,

$$T = -\frac{W}{\lambda} \cos \lambda s + \frac{X}{\lambda} \sin \lambda s + Y.$$

Integrating this, we get

$$\beta(s) = -\frac{W}{\lambda^2} \sin \lambda s - \frac{X}{\lambda^2} \cos \lambda s + Y s + Z,$$

where W, X, Y, Z are any three-dimensional vectors which satisfy $|T| = 1$.

The circular helix (Figure 1.2) is obtained by letting W, X, Y and Z to be $(1,0,0)$, $(0,1,0)$, $(0,0,\sqrt{1-1/\lambda^2})$ and $(0,0,0)$ respectively. We then have

$$\beta(s) = \left(-\frac{1}{\lambda^2} \sin \lambda s, -\frac{1}{\lambda^2} \cos \lambda s, \sqrt{(1-1/\lambda^2)}\, s\right).$$

4.3 Simplest Curves in Four-dimensions

A simplest curve in four dimensions should be one with κ, τ and σ constant and non-zero. The Frenet formulae in four dimensions are

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N + \sigma D, \quad \frac{dD}{ds} = -\sigma B.$$

We differentiate the first equation continuously until we get an equation involving the vector T only.

$$\frac{d^2T}{ds^2} = \kappa \frac{dN}{ds} = -\kappa^2 T + \kappa\tau B. \tag{6}$$

$$\begin{aligned}\frac{d^3T}{ds^3} &= -\kappa^2 \frac{dT}{ds} + \kappa\tau \frac{dB}{ds} \\ &= -\kappa^2 \frac{dT}{ds} + \kappa\tau(-\tau N + \sigma D) \\ &= -\kappa^2 \frac{dT}{ds} - \tau^2 \frac{dT}{ds} + \kappa\tau\sigma D. \\ \frac{d^4T}{ds^4} &= -(\kappa^2 + \tau^2)\frac{d^2T}{ds^2} + \kappa\tau\sigma \frac{dD}{ds} = -(\kappa^2 + \tau^2)\frac{d^2T}{ds^2} - \kappa\tau\sigma^2 B \\ &= -(\kappa^2 + \tau^2)\frac{d^2T}{ds^2} - \sigma^2 \left(\frac{d^2T}{ds^2} + \kappa^2 T\right) \qquad \text{(from(6))} \\ &= -(\kappa^2 + \tau^2 + \sigma^2)\frac{d^2T}{ds^2} - \kappa^2\sigma^2 T. \end{aligned} \tag{7}$$

Let $T = V \cos \lambda s + W \sin \lambda s$ where V and W are arbitrary constant vectors. Then

$$\frac{d^2T}{ds^2} = -\lambda^2 (V \cos \lambda s + W \sin \lambda s) = -\lambda^2 T, \tag{8}$$

$$\frac{d^4T}{ds^4} = -\lambda^2(-\lambda^2 T) = \lambda^4 T \tag{9}$$

Substituting (8) and (9) into (7), we have

$$\lambda^4 T - (\kappa^2 + \tau^2 + \sigma^2)\lambda^2 T + \kappa^2\sigma^2 T = 0.$$

Since $T \neq 0$, we have the following quartic polynomial equation

$$\lambda^4 - (\kappa^2 + \tau^2 + \sigma^2)\lambda^2 + \kappa^2\sigma^2 = 0, \tag{10}$$

which can be reduced to a quadratic equation in λ^2. Solving, we have

$$\lambda^2 = \frac{1}{2}\left(\kappa^2 + \tau^2 + \sigma^2 \pm \sqrt{(\kappa^2 + \tau^2 + \sigma^2)^2 - 4\kappa^2\sigma^2}\right). \tag{11}$$

The discriminant

$$(\kappa^2 + \tau^2 + \sigma^2)^2 - 4\kappa^2\sigma^2 = (\kappa^2 + \tau^2 - \sigma^2)^2 + 4\tau^2\sigma^2 > 0$$

since $\kappa, \tau, \sigma \neq 0$. Hence λ^2 has two real and unequal values. Furthermore

$$\kappa^2 + \tau^2 + \sigma^2 > \sqrt{(\kappa^2 + \tau^2 + \sigma^2)^2 - 4\kappa^2\sigma^2}.$$

Hence λ^2 has two real, unequal and positive values. λ can therefore take four values, λ_0, $-\lambda_0$, λ_1, $-\lambda_1$.

When $\lambda = -\lambda_0$,

$$\begin{aligned} T &= V\cos(-\lambda_0 s) + W\sin(-\lambda_0 s) \\ &= V\cos\lambda_0 s + (-W)\sin\lambda_0 s. \end{aligned}$$

Since V and W are arbitrary, the solution is similar to that where $\lambda = \lambda_0$. Therefore, we need only consider two values of $\lambda : \lambda_0$ and λ_1.

Let

$$\begin{aligned} T_0 &= V\cos\lambda_0 s + W\sin\lambda_0 s, \\ T_1 &= X\cos\lambda_1 s + Y\sin\lambda_1 s. \end{aligned}$$

Since T_0 and T_1 are two solutions of the differential equation (7), we get

$$\frac{d^4}{ds^4}(T_0 + T_1) + (\kappa^2 + \tau^2 + \sigma^2)\frac{d^2}{ds^2}(T_0 + T_1) + \kappa^2\sigma^2(T_0 + T_1) = 0.$$

Thus $T_0 + T_1$ is also a solution of the differential equation. By the theory of differential equations, there are no other solutions.

Integrating $T_0 + T_1$, we have

$$\beta(s) = \frac{V}{\lambda_0}\sin\lambda_0 s - \frac{W}{\lambda_0}\cos\lambda_0 s + \frac{X}{\lambda_1}\sin\lambda_1 s - \frac{Y}{\lambda_1}\cos\lambda_1 s + Z.$$

Take $V = \sqrt{\frac{1}{2}}(1,0,0,0)$, $W = \sqrt{\frac{1}{2}}(0,1,0,0)$, $X = \sqrt{\frac{1}{2}}(0,0,1,0)$, $Y = \sqrt{\frac{1}{2}}(0,0,0,1)$, $Z = (0,0,0,0)$, and we get the following curve:

$$\beta(s) = \sqrt{\frac{1}{2}}\left(\frac{1}{\lambda_0}\sin\lambda_0 s, -\frac{1}{\lambda_0}\cos\lambda_0 s, \frac{1}{\lambda_1}\sin\lambda_1 s, -\frac{1}{\lambda_1}\cos\lambda_1 s\right). \quad (12)$$

With this choice of V, W, X, Y and Z, T is automatically a unit vector.

4.4 Properties of the Four-dimensional Simplest Curve

In this section, we examine some of the more interesting properties of the curve described in equation (12). λ_0^2 and λ_1^2 are the two solutions for λ^2 in equation (10). Now $\lambda_0^2 + \lambda_1^2$ is the sum of roots and $\lambda_0^2\lambda_1^2$ is the product of roots. Hence,

$$|\beta| = \sqrt{\frac{1}{2\lambda_0^2} + \frac{1}{2\lambda_1^2}} = \sqrt{\frac{\lambda_0^2 + \lambda_1^2}{2\lambda_0^2\lambda_1^2}}$$

$$= \sqrt{\frac{\kappa^2 + \tau^2 + \sigma^2}{2\kappa^2\sigma^2}} = \sqrt{\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{\tau^2}{\kappa^2\sigma^2} + \frac{1}{\kappa^2}\right)}.$$

Since κ, τ and σ are all constant, we see that the curve lies on a four-dimensional sphere. When κ or σ increases, then $|\beta|$ decreases, hence the sphere is smaller. However when τ increases, the sphere becomes larger. When $\kappa = \tau = \sigma$, $|\beta| = \frac{1}{\kappa}\sqrt{\frac{3}{2}}$.

Although it is impossible for us to have a true picture of what the curve is actually like in four dimensions, we can try to have a glimpse by examining the curve with one dimension 'cut off'. Since $\beta(s)$ is given by (12), cutting off any dimension will produce the same type of curve. Let's cut off the last dimension, to obtain

$$\alpha(s) = \left(\frac{1}{\lambda_0}\sin\lambda_0 s, -\frac{1}{\lambda_0}\cos\lambda_0 s, \frac{1}{\lambda_1}\sin\lambda_1 s\right).$$

A rough sketch of the curve is shown in Figure 4.1 resembling a spring with both ends packed together. The derivation of this sketch is as follows. The x and y coordinates correspond to a circle. The z coordinate is a sine function which means that a particle moving along the curve travels faster in the middle than at the ends.

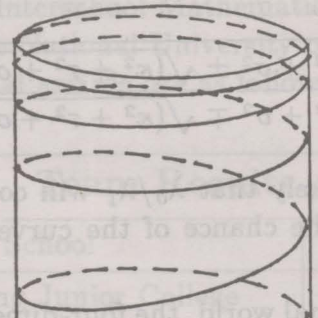

Figure 4.1

An interesting question is whether the original curve repeats itself. For it to repeat itself, there must exist a constant a such that

$$\sqrt{\frac{1}{2}}\begin{pmatrix} \frac{1}{\lambda_0}\sin\lambda_0 s \\ -\frac{1}{\lambda_0}\cos\lambda_0 s \\ \frac{1}{\lambda_1}\sin\lambda_1 s \\ -\frac{1}{\lambda_1}\cos\lambda_1 s \end{pmatrix} = \sqrt{\frac{1}{2}}\begin{pmatrix} \frac{1}{\lambda_0}\sin\lambda_0 (s+a) \\ -\frac{1}{\lambda_0}\cos\lambda_0 (s+a) \\ \frac{1}{\lambda_1}\sin\lambda_1 (s+a) \\ -\frac{1}{\lambda_1}\cos\lambda_1 (s+a) \end{pmatrix}$$

Comparing the first component, we have

$$\lambda_0 s = \begin{cases} (2m+1)\pi - \lambda_0 s - \lambda_0 a & (13) \\ 2m\pi + \lambda_0 s + \lambda_0 a. & (14) \end{cases}$$

Comparing the second component, we have

$$\lambda_0 s = \begin{cases} 2n\pi - \lambda_0 s - \lambda_0 a & (15) \\ 2n\pi + \lambda_0 s + \lambda_0 a. & (16) \end{cases}$$

where m, n are integers. (13) and (15) cannot be true for all s. So (14) and (16) must be true which means that $-\lambda_0 a = 2m\pi$. Similarly, by comparing the third and the fourth components, we get $-\lambda_1 a = 2n\pi$. Thus, $\lambda_0/\lambda_l = m/n$. Since m and n are integers, the value λ_0/λ_1 must be a rational number in order for the curve to repeat itself. Referring to (11), we find that

$$\frac{\lambda_0}{\lambda_1} = \sqrt{\frac{\kappa^2 + \tau^2 + \sigma^2 \pm \sqrt{(\kappa^2 + \tau^2 + \sigma^2)^2 - 4\kappa^2\sigma^2}}{\kappa^2 + \tau^2 + \sigma^2 \mp \sqrt{(\kappa^2 + \tau^2 + \sigma^2)^2 - 4\kappa^2\sigma^2}}}.$$

We can see that it is unlikely that λ_0/λ_1 will come out to be a rational value. This means that the chance of the curve repeating itself is very small.

In our three-dimensional world, the four-dimensional Frenet formulae may seem irrelevant and useless. However, this is not so. The Frenet formulae have been applied in many areas of Physics. This is possible because we can let one of the axes in four dimensions to be time. Thus, this three-dimensional time graph shows the motion of a particle in three dimensions. The Frenet formulae can therefore be used to describe this motion. In "Classical and Quantum Gravity," Vol. 5, No. 7, there is a research paper written by B R Iyer and C V Vishveshwara whose title is 'The Frenet-Serret Formalism and Black Holes in Higher Dimensions'. The paper examines the motion of charged particles in homogenous electromagnetic fields using the Frenet formulae for dimensions higher than three.

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