

Basic Concepts in Hypothesis Testing*

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§1. Introduction

Most people make rough confidence statements on many aspects of their daily lives by the choice of an adjective, an adverb, or a phrase. Scientific investigators, industrial engineers and market researchers, among others, often have hypotheses about particular facets of their work areas. These hypotheses need substantiation or verification — or rejection — for one purpose or another. To this end, they gather data and allow them to support or cast doubt on their hypotheses. This process is termed *hypothesis testing* and is one of two main areas of statistical inference.

Definition : A *statistical hypothesis* is an assertion or conjecture about the distribution of one or more random variables. If a statistical hypothesis completely specifies the distribution, it is referred to as a *simple hypothesis*; if not, it is referred to as a *composite hypothesis*.

§2. General Steps in Hypothesis Testing

The testing of a statistical hypothesis is the application of an explicit set of rules for deciding whether to accept the hypothesis or to reject it. The method of conducting any statistical hypothesis testing can be outlined in six steps :

1. Decide on the null hypothesis H_0

The *null hypothesis* generally expresses the idea of *no difference*. The symbol we use to denote a null hypothesis is H_0 .

2. Decide on the alternative hypothesis H_1

The *alternative hypothesis*, which we denote by H_1 , expresses the idea of *some difference*. Alternative hypotheses may be one-sided or two-sided. Usually the setting of the problem determines the alternative even before the data has been collected.

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3. Calculate the appropriate test statistic

This is a value that we will calculate from the sample data.

4. Decide on the significance level or the critical P -value

All hypothesis testing is liable to errors. There are two basic kinds of error :

- Type I error : Reject H_0 when it is, in fact, true; the probability of committing a type I error is denoted by α .
- Type II error : Reject H_1 when it is, in fact, true; the probability of committing a type II error is denoted by β .

The objective in all hypothesis testing is to set the Type I error level, also known as the *significance level*, at a low enough value, and then to use a *test statistic* which minimises the Type II error level for a given sample size. As we fix the Type I error level, it is best to devise the test in such a way that the Type I error is most serious, in terms of cost.

A *critical P-value* is the probability that is set by the person doing the test; it is the threshold for the P -value that the tester will use to decide whether the sample is unusual enough, compared to the hypothesised population, to indicate that the null hypothesis should be rejected in favour of the alternative.

5. The P -value or critical region of size α

The calculated test statistic is compared to the sampling distribution that the statistic would have if the null hypothesis were true. The comparison is summarised into a probability called a P -value : this is the probability, if the null hypothesis is true, that the statistic would be at least as far from the expected value as it was observed to be in the sample.

The P -value ranges from 0.0 to 1.0. As it approaches 0.0, it indicates that the sample is a rare outcome if the population is as hypothesized. The closer the P -value is to zero, the stronger the evidence against the null hypothesis.

When we are testing the null hypothesis $H_0 : \theta = \theta_0$ against the two-sided alternative hypothesis $H_1 : \theta \neq \theta_0$, the critical region consists of both tails of the sampling distribution of the test statistic. Such a test is a *two-tailed test*. On the other hand, if we are testing the null hypothesis $H_0 : \theta = \theta_0$ against one-sided alternative $H_1 : \theta < \theta_0$ or $H_1 : \theta > \theta_0$, the critical regions are the *left tail* or *right tail* of the sampling distribution of the test statistic respectively.

6. Statement of conclusion

A decision is made based on the size of the P -value. When the P -value is small (i.e. less than the critical P -value), we reject the null hypothesis. When it is not small (greater than the critical P -value), we accept the null hypothesis.

In the same way, if the value of the test statistic falls in the critical region, we reject the null hypothesis.

The conclusion should, as far as possible, be devoid of statistical terminology. However the significance level should be stated.

The assumption of this test is that the variable is approximately normally distributed. This assumption is less critical the larger the sample size.

In the next section we will explain this hypothesis testing procedure in a variety of situations. The criteria that distinguish between the procedures are :

- (a) the statistic used to summarise the sample. This in turn depends on the hypothesis to be tested.
- (b) the size of the sample
- (c) knowledge about the population before the sampling is done
- (d) the number of populations sampled
- (e) the number of variables observed

§3. Applications of Hypothesis Testing

In this section, some of the standard tests that are most widely used in applications are presented. All the tests presented are based on normal distribution theory, assuming either that the samples are from normal populations or that they are large enough to justify normal approximations.

3.1 Tests concerning Averages

Suppose we want to test the null hypothesis $\mu = \mu_0$ against one of the alternatives $\mu \neq \mu_0$, $\mu > \mu_0$, or $\mu < \mu_0$ on the basis of a random sample

sample of size n from the normal population with known variance σ^2 . The appropriate test statistic for this test is

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}.$$

The critical regions for the respective alternatives are $|z| \geq z_{\alpha/2}$, $z \geq z_{\alpha}$, and $z \leq -z_{\alpha}$.

Example : Suppose that it is known from experience that the standard deviation of the weight of 8-ounce packages of cookies made by a certain bakery is 0.16 oz. To check whether its production is under control on a given day, they select a random sample of 25 packages and find that their mean weight is $\bar{x} = 8.112$ oz. Since the bakery stands to lose money when $\mu > 8$ and the customer loses out when $\mu < 8$, test the null hypothesis $\mu = 8$ against the alternative $\mu \neq 8$ using $\alpha = 0.01$.

Solution

1. Null hypothesis : $H_0 : \mu = 8$
2. Alternative hypothesis : $H_1 : \mu \neq 8$
3. Test statistic : $z = \frac{8.112 - 8}{0.16/\sqrt{25}} = 3.50$
4. Significance level : 1 %
5. Critical regions : Two-tailed test. Reject H_0 if $z \leq -2.575$ or $z \geq 2.575$
6. Conclusion : Since $z = 3.50$ exceeds 2.575, reject the null hypothesis at the 1% significance level.

When dealing with a large sample of size $n \geq 30$ from a population which need not be normal but has finite variance, we can use the central limit theorem to justify using the test for normal populations. Even when σ^2 is unknown we can approximate its value with s^2 in the computation of the test statistic.

When $n < 30$ and σ^2 is unknown, the above test cannot be used. Instead, a *one sample t-test* is used. This test is based on the following test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}},$$

which is a value of a random variable having the t distribution with $n - 1$ degrees of freedom. The critical regions of size α for testing the null hypothesis $\mu = \mu_0$ against the three alternatives $\mu \neq \mu_0$, $\mu > \mu_0$, and $\mu < \mu_0$ are $|t| \geq t_{\alpha/2, n-1}$, $t \geq t_{\alpha, n-1}$ and $t \leq -t_{\alpha, n-1}$, respectively.

Example Suppose that the specifications for a certain kind of ribbon call for a mean breaking strength of 185 pounds, and that 5 pieces randomly selected from different rolls have a mean breaking strength of 183.1 pounds with a standard deviation of 8.2 pounds. Assuming that we can look upon the data as a random sample from a normal population, test the null hypothesis $\mu = 185$ against the alternative hypothesis $\mu < 185$ at $\alpha = 0.05$.

Solution

1. Null hypothesis : $H_0 : \mu = 185$
2. Alternative hypothesis : $H_1 : \mu < 185$
3. Test statistic : $t = \frac{183.1 - 185}{8.2\sqrt{5}} = -0.49$
4. Significance level : 5 %
5. Critical region : One-tailed test. Reject H_0 if $t \leq -2.132$
6. Conclusion : Since $t = -0.49$ is greater than -2.132 , the null hypothesis cannot be rejected.

3.2 Tests concerning Differences between Means

Suppose we have independent random samples of size n_1 and n_2 from two normal populations having means μ_1 and μ_2 and known variances σ_1^2 and σ_2^2 , respectively, and that we want to test the null hypothesis $\mu_1 - \mu_2 = \delta$, where δ is a given constant, against one of the alternatives $\mu_1 - \mu_2 \neq \delta$, $\mu_1 - \mu_2 > \delta$, or $\mu_1 - \mu_2 < \delta$. The appropriate test statistic for this test is

$$z = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{\sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}}$$

The critical regions for the respective alternatives are $|z| \geq z_{\alpha/2}$, $z \geq z_\alpha$, and $z \leq -z_\alpha$.

When we deal with independent random samples from populations with unknown variances which may not even be normal, the test can still be used with s_1 substituted for σ_1 and s_2 substituted for σ_2 so long as both samples are large enough for the central limit theorem to be invoked.

Example Suppose that the nicotine contents of two brands of cigarettes are being measured. If, in an experiment fifty cigarettes of the first brand had an average nicotine content of $\bar{x}_1 = 2.61mg$ with a standard deviation of $s_1 = 0.12mg$, while forty cigarettes of the second brand had an average nicotine content of $\bar{x}_2 = 2.38mg$ with a standard deviation of $s_2 = 0.14mg$, test the null hypothesis $\mu_1 - \mu_2 = 0.2$ against the alternative $\mu_1 - \mu_2 \neq 0.2$, using $\alpha = 0.05$.

Solution

1. Null hypothesis : $H_0 : \mu_1 - \mu_2 = 0.20$
2. Alternative hypothesis : $H_1 : \mu_1 - \mu_2 \neq 0.20$
3. Test statistic : $z = \frac{2.61 - 2.38 - 0.2}{\sqrt{[(0.12)^2/50] + [(0.14)^2/40]}} = 1.08$
4. Significance level : 5 %
5. Critical region : Two-tailed test. Reject H_0 if $z \leq -1.96$ or $z \geq 1.96$
6. Conclusion : Since $z = 1.08$ falls between -1.96 and 1.96 , we cannot reject the null hypothesis.

When n_1 and n_2 are small and σ_1 and σ_2 are unknown, we cannot use the above test. However, for independent random samples from two normal populations having the same unknown variance σ^2 , we use the two-sample t test. The appropriate test statistic for this test is

$$t = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{s_p \sqrt{(1/n_1) + (1/n_2)}}$$

where

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

The respective critical regions are $|t| \geq t_{\alpha/2, n_1+n_2-2}$, $t \geq t_{\alpha, n_1+n_2-2}$, and $t \leq -t_{\alpha, n_1+n_2-2}$.

3.3 Tests concerning Differences between Means of Paired Observations

In this case the samples are not independent because the data are actually paired. The way of handling this is to look upon the differences as a random sample of size n from a normal population which, under the null hypothesis, has mean $\mu = \delta$. Then we test this null hypothesis against

the appropriate alternative by means of the methods described in Section 3.1. The appropriate test statistic depends on the size of n .

Example An experimenter studied the effect of exposure of lucerne flowers to different environmental conditions. He chose 10 vigorous plants with freely exposed flowers at the top and flowers hidden as much as possible at the bottom. Finally he determined the number of seeds set per two pods at each location. The data were :

Plant	1	2	3	4	5	6	7	8	9	10
Top flowers	4.8	5.2	5.7	4.2	4.8	3.9	4.1	3.0	4.6	6.8
Bottom flowers	4.4	3.7	4.7	2.8	4.2	4.3	3.5	3.7	3.1	1.9

Test the hypothesis of no difference between population means against the alternative that top flowers set more seeds for $\alpha = 0.05$.

Solution The sample of differences (top flowers - bottom flowers) is (0.4, 1.5, 1.0, 1.4, 0.6, -0.4, 0.6, -0.7, 1.5, 4.9). The estimated mean $\bar{x} = 1.08$ and standard deviation $s = 1.54$.

1. Null hypothesis : $H_0 : \mu = 0$
2. Alternative hypothesis : $H_1 : \mu > 0$
3. Test statistic : $t = \frac{1.08 - 0}{1.54/\sqrt{10}} = 2.22$
4. Significance level : 5 %
5. Critical region : One-tailed test. Reject H_0 if $t \geq 2.132$
6. Conclusion : Since $t = 2.22$ is greater than 2.132, the null hypothesis cannot be rejected.

References

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