The Reflection Principle and Hearing the Shape of a Drum*

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1. Introduction.

In a beautiful article Kac (1966) asked whether knowledge of the eigenvalues of the Laplacian operating on smooth functions defined on a bounded region $\mathcal{R}$ of the plane and vanishing on the (assumed smooth) boundary, $\partial\mathcal{R}$, uniquely determines the geometry of $\mathcal{R}$. If one interprets the eigenvalues as normal modes of vibration of a thin membrane which is held fixed by a wire frame forming the boundary and which vibrates in the region $\mathcal{R}$ in accordance with the wave equation, the question can be phrased in physical terms: "Can one hear the shape of a drum?" Kac provided a partial solution by probabilistic methods.

In this paper I discuss, also by probability methods, the expansion

$$(1)\quad \Sigma \exp(-\lambda_k t) = (2\pi t)^{-1}|\mathcal{R}| - [4(2\pi t)^{\frac{1}{2}}]^{-1}|\partial\mathcal{R}| + (1-h)/6 + 2^{-8}(2\pi)^{-\frac{1}{2}}\left(\int_{\partial\mathcal{R}} c^2(s)ds\right)t^{\frac{1}{2}} + o(t^{\frac{1}{2}}),$$

where $\lambda_1 < \lambda_2 < \ldots$ are the eigenvalues of the Laplacian, $|\mathcal{R}|$ denotes the area of $\mathcal{R}$, $|\partial\mathcal{R}|$ is the length of the boundary, h is the number of holes in $\mathcal{R}$, s is arc length on the boundary, and $c(s)$ is the curvature of the boundary at s. The first two terms were obtained by Kac, who also conjectured the third after a lengthy heuristic calculation. Subsequent authors have used analytic methods to obtain even more detailed expansions. Some references are given by Lerche and Siegmund (1987). With respect to Kac's original question, one can say on the basis of (1) that one can "hear" the area of a drum, the length of its boundary, and the number of holes. A definitive answer to Kac's question remains unknown, although the answer to the analogous mathematical question in more than three dimensions is known to be negative.

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2. Random Walk, the Reflection Principle, and Brownian Motion.

In this section I give the probabilistic background relevant to a proof of (1). In order to make the presentation elementary insofar as possible, I begin with simple random walk, derive a result equivalent to the so called reflection principle, and finally introduce the more technical subject of Brownian motion. See Feller (1957, Chapter III) for a classical treatment of the reflection principle.

Let $x_1, x_2, \ldots$ be independent random variables assuming the values of $+1$ and -1 with probability $1/2$ each, and put $S_n = x_1 + \ldots + x_n$. Let

$$f_n^{(m)}(\delta_1, \ldots, \delta_n | \xi) = \Pr(x_1 = \delta_1, \ldots, x_n = \delta_n | S_m = \xi).$$

Note that $\Pr(S_n = k) = \binom{n}{(n+k)/2} 2^{-n}$ and hence

$$(2) \qquad f_n^{(m)}(\delta_1, \ldots, \delta_n | \xi) = \frac{\Pr(x_1 = \delta_1, \ldots, x_n = \delta_n) \Pr(S_m - S_n = \xi - \sum_1^n \delta_i)}{\Pr(S_m = \xi)}$$

$$= \binom{m-n}{(m-n+\xi-\sum_1^n \delta_i)/2} \Big/ \binom{m}{(m+\xi)/2},$$

provided the denominator is positive, and 0 otherwise.

Let b be a positive integer and $\tau = \inf\{n : S_n = b\}$, with the understanding that $\inf(\phi) = +\infty$.

Proposition 1. Let $\xi < b$ be an arbitrary integer, and let ξ' be any other integer such that $\xi + \xi'$ is even. Then

$$\Pr(\tau < m | S_m = \xi) = E\{1_{(\tau < m)} R_\tau | S_m = \xi'\},$$

where $R_n = f_n^{(m)}(x_1, \ldots, x_n | \xi) / f_n^{(m)}(x_1, \ldots, x_n | \xi')$, and $0/0$ is interpreted as 0.

Proof. Obviously $\Pr(\tau < m | S_m = \xi)$

$$= \sum_{n=1}^{m-1} \Pr(\tau = n | S_m = \xi) = \sum_{n=1}^{m-1} \sum_{\{(\delta_1, \ldots, \delta_n) : \tau = n\}} f_n^{(m)}(\delta_1, \ldots, \delta_n | \xi).$$

Dividing and multiplying by $f_n^{(m)}(\delta_1,\ldots,\delta_n|\xi')$, one obtains

$$\begin{aligned}\Pr(\tau < m|S_m = \xi) &= \sum_{n=1}^{m-1} E\{1_{(\tau=n)} R_n|S_m = \xi'\} \\ &= E(1_{(\tau<m)} R_\tau|S_m = \xi')\end{aligned}$$

Corollary. For arbitrary $\xi < b$

$$\Pr(\tau < m|S_m = \xi) = \binom{m}{(m+2b-\xi)/2} \Big/ \binom{m}{(m+\xi)/2}.$$

Proof. Let $\xi' = 2b - \xi > b$. Since $S_\tau = b$ whenever $\tau < m$ and $\binom{m}{(m+i)/2} = \binom{m}{(m-i)/2}$, it follows from (2) that

$$R_\tau = \binom{m}{(m+2b-\xi)/2} \Big/ \binom{m}{(m+\xi)/2}$$

whenever $\tau < m$. Since $\xi' > b$, $\Pr(\tau < m|S_m = \xi') = 1$, and hence the corollary follows immediately from proposition 1.

Remark. The classical combinatorial proof of the corollary is based on the observation that the number of possible paths, S_n, $n = 0,1,\ldots,m$, from $(0,0)$ to (m,ξ) which touch the line $y = b$ equals the total number of paths from $(0,0)$ to $(m, 2b-\xi)$. The virtue of the preceding derivation is that the identity in Proposition 1 is completely general, i.e. it does not depend on the exact definition of the first passage time τ. Hence in perturbations of the present problem where simple exact solutions do not exist, there is the possibility of analyzing Proposition 1 to obtain approximate solutions.

A Brownian motion or Wiener process, $w(t)$, is in a sense which cannot be made precise here the limit as $m \to \infty$ of $S_{[mt]}/m^{1/2}$. Corresponding to Proposition 1, we have for $b > 0$ and $\xi < b$

$$\begin{aligned}&(3)\ \Pr(\tilde{\tau} < t|w(t) = \xi) \\ &\quad = E\left\{1_{(\tilde{\tau}<t)} \exp\left[(\xi - \xi')\{w(\tilde{\tau}) - (\xi + \xi')\tilde{\tau}/2t\}/(t - \tilde{\tau})\right]|w(t) = \xi'\right\},\end{aligned}$$

where $\tilde{\tau} = \inf\{s : w(s) = b\}$. Putting $\xi' = 2b - \xi$, one easily obtains from (3)

$$(4) \qquad \Pr(\tilde{\tau} < t | w(t) = \xi) = \exp[-2b(b - \xi)/t].$$

It is possible to derive (3) from Proposition 1 by a brute force calculation. A more elegant approach uses the optional sampling theorem of martingale theory.

Remark. An analytic approach to the results of this section is to set

$$u_b(m, \xi) = \Pr\{\tau > m, S_m = \xi\}\big(= \Pr\{S_m = \xi\}[1 - \Pr\{\tau < m | S_m = \xi\}]\big)$$

and observe that $u = u_b$ satisfies

$$u(m+1, \xi) = \frac{1}{2}\, u(m, \xi - 1) + \frac{1}{2}\, u(m, \xi + 1) \qquad (\xi < b)$$

with the initial and boundary conditions $u(0,0) = 1, u(m, b) = 0$. Putting

$$\tilde{u}(t, \xi) = \lim_{m \to \infty} u_{m^{\frac{1}{2}} b}([mt], [m^{\frac{1}{2}} \xi]),$$

one can show that

$$\tilde{u}(t, \xi) d\xi = \Pr\{\tilde{\tau} > t, w(t) \in d\xi\}$$

and $\tilde{u}$ satisfies the heat equation,

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial \xi^2} \qquad (\xi < b),$$

with the initial and boundary conditions $\tilde{u}(0,0) =$ Dirac delta function, $\tilde{u}(t, b) = 0$. Although we do not solve these equations here, the relation of Brownian motion to the heat equation is at the foundation of the developments in the next section. At first it seems remarkable that one can learn something about wave motion from the study of heat flow, or diffusion, but the eigenvalue problems are the same.

3. Brownian Motion and Hearing the Shape of a Drum.

Let $\mathcal{R}$ be a bounded region in the plane with a smooth boundary $\partial\mathcal{R}$. Let $\xi_0 \in \mathcal{R}$ and let $w(t)$ be a two–dimensional Brownian motion process (a vector of two independent one–dimensional Brownian motions) with $w(0) = \xi_0$. Let $T = \inf\{t : w(t) \notin \mathcal{R}\}$ and define $p(t, \xi_0, \xi_1)$ by

$$p(t, \xi_0, \xi_1)d\xi_1 = \Pr(T > t, w(t) \in d\xi_1 | w(0) = \xi_0).$$

It is known that $p(t, \xi_0, \xi_1)$ satisfies the heat equation $\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p$ and equals 0 on $\partial\mathcal{R}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian and (x, y) can be taken to be either ξ_0 or ξ_1. It follows from general theory of the heat equation that $p(t, \xi_0, \xi_1)$ has an expansion of the form $p(t, \xi_0, \xi_1) = \Sigma e^{-\lambda_k t}\varphi_k(\xi_0)\varphi_k(\xi_1)$, where the λ_k are eigenvalues of $\frac{1}{2}\Delta$ and $\{\varphi_n, n = 1, 2, \ldots\}$ is a complete orthonormal set of eigenfunctions vanishing on $\partial\mathcal{R}$. As a consequence

$$\Sigma \exp(-\lambda_k t) = \iint_{\mathcal{R}} p(t, \xi_0, \xi_0)d\xi_0.$$

From this and the relations

$$\begin{aligned} p(t, \xi_0, \xi_1)d\xi_1 &= \Pr(w(t) \in d\xi_1 | w(0) = \xi_0)\{1 - \Pr(T < t | w(0) = \xi_0, w(t) = \xi_1)\} \\ &= (2\pi t)^{-1} \exp\{- \| \xi_1 - \xi_0 \|^2 /2t\} d\xi_1 [1 - q(t, \xi_0, \xi_1)], \end{aligned}$$

where we have put $q(t, \xi_0, \xi_1) = \Pr(T < t | w(0) = \xi_0, w(t) = \xi_1)$, follows

$$\Sigma \exp(-\lambda_k t) = (2\pi t)^{-1} \left\{ |\mathcal{R}| - \iint_{\mathcal{R}} q(t, \xi_0, \xi_0)d\xi_0 \right\}. \tag{5}$$

Since $q(t, \xi_0, \xi_0) \to 0$ as $t \to 0$ for each $\xi_0 \in \mathcal{R}$, the first term in (1) follows at once from (5). To obtain the higher order terms in (1) one must evaluate $q(t, \xi_0, \xi_0)$ for $t \to 0$ and ξ_0 approaching $\partial\mathcal{R}$.

To this end consider the point on $\partial\mathcal{R}$ closest to ξ_0 and the cartesian coordinate system having this point as its origin, the outward normal as its y–axis and the tangent to $\mathcal{R}$ as its x–axis. Locally in a neighborhood of this point $\partial\mathcal{R}$ is given by the graph of a function $y = f(x)$, where $f(0) = f'(0) = 0$. Let $w(t)$ have coordinates $w_1(t)$ and $w_2(t)$ in this tangent–normal coordinate system and define

$$T^* = \inf\{t : w_2(t) \geq f(w_1(t))\}.$$

Let $q^*(t,\xi_0) = \Pr(T^* < t | w(0) = \xi_0, w(t) = \xi_0)$. If q is replaced by q^* in (4) the error is exponentially small as $t \to 0$, and hence it suffices to study the asymptotic behavior of $q^*(t,\xi_0)$ as $t \to 0$ and $\xi_0 = (0,-y_0) \to (0,0)$. Let $\xi_0' = (0,y_0)$. By a slight extension of (3) (with $b = y_0, \xi = 0, \xi' = 2y_0$, and $w(\tilde{\tau})$ replaced by $w_2(T^*) - y_0$)

(6) $q^*(t,\xi_0)$

$$= e^{-2y_0^2/t} E\left\{\exp\left[-2y_0 w_2(T^*)/(t-T^*)\right] \middle| w(0) = \xi_0, w(t) = \xi_0'\right\}$$

$$= e^{-2y_0^2/t} E\left\{\exp\left[-2y_0 f(w_1(T^*))/(t-T^*)\right] \middle| w(0) = \xi_0, w(t) = \xi_0'\right\}.$$

The rest of the proof of (1) is a detailed analysis of (6), which is then substituted for q in (5) for ξ_0 close to $\partial\mathcal{R}$ and integrated. To see what is involved without considering the substantial technical details, note that since $f(x) \sim x^2 f''(0)/2$ as $x \to 0$ and $w_1^2(T^*)$ is with overwhelming probability of order T^*, which in turn is $< t$, the argument of the exponential function inside the conditional expectation in (6) is of order $y_0 \to 0$ as $t \to 0$. Hence

$$(7) \qquad q^*(t,\xi_0) \sim e^{-2y_0^2/t} \qquad (t, y_0 \to 0),$$

which says in effect that when ξ_0 is close to $\partial\mathcal{R}$ the probability of crossing $\partial\mathcal{R}$ is to a first approximation the same as the probability of crossing the tangent at the point closest to ξ_0 (cf. (4) with $\xi = 0$).

A change of variable in (5) and some simple estimates show that for sufficiently small ε

$$(8) \qquad \iint_{\mathcal{R}} q(t,\xi_0,\xi_0)d\xi_0 = \int_{\partial\mathcal{R}}\int_0^{\varepsilon} q^*(t,\xi_0)[1 - y_0 c(s)]dy_0\, ds + O(e^{-2\varepsilon^2/t}),$$

where s is arc length on $\partial\mathcal{R}$ and $c(s)$ is the curvature of $\partial\mathcal{R}$ at s. Substitution of (7) into (8) easily yields the second term in (1).

To obtain the third term in (1) we use a Taylor series expansion to show that the conditional expectation on the right hand side of (6) equals

$$(9) \qquad 1 - y_0 f''(0) E\left\{w_1^2(T^*)/(t-T^*) | w(0) = \xi_0, w(t) = \xi_0'\right\} + \ldots .$$

Moreover, in the limit as $t, y_0 \to 0$, T^* may be replaced by $\tau = \inf\{t : w_2(t) \geq 0\}$. This substitution and some fairly standard calculation show that (9) equals

$$1 - y_0 f''(0) E\left\{w_1^2(\tau)/(t-\tau) | w(0) = \xi_0, w(t) = \xi_0'\right\} + \ldots$$

$$= 1 - f''(0) t^{-\frac{1}{2}} y_0^2 \Phi(-2y_0/t^{\frac{1}{2}})/\varphi(2y_0/t^{\frac{1}{2}}) + \ldots,$$

where $\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(u)du$. If the boundary $\partial\mathcal{R}$ is parameterized by arc length s and if the origin $(0,0)$ of our special tangent–normal coordinate system corresponds to s_0, then $f''(0) = -c(s_0)$, where c is the curvature of $\partial\mathcal{R}$. Hence by (6)

$$q^*(t, \xi_0) = e^{-2y_0^2/t}[1 + c(s_0)t^{-\frac{1}{2}}\ y_0^2 \Phi(-2y_0/t^{\frac{1}{2}})/\varphi(2y_0/t^{\frac{1}{2}}) + \ldots] ,$$

which when substituted into (8) yields the second and third terms of (1). (Note that $\int_{\partial\mathcal{R}} c(s)ds = 2\pi(1-h)$.)

The pattern is now clear, although admittedly the details are not. To obtain the fourth term in (1), one must take one more term of the Taylor series in (9). It is necessary to distinguish between the cases $f''(0) < 0$ and $f''(0) > 0$ in order to approximate the difference between T^* and τ; and with substantially more calculation one can complete the proof of (1). Details of this argument, related results, and additional applications are given by Lerche and Siegmund (1987).

References

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Biographical Note. David Siegmund received his Ph.D. in mathematical statistics from Columbia University in 1966. Since then he has taught at Columbia and at Stanford University, where he is presently Professor of Statistics. He has also been a visiting professor at the Hebrew University of Jerusalem, the University of Zürich, and Heidelberg University. Professor Siegmund is also an external examiner for the Mathematics Department, National University of Singapore (1985–87 and 1987–89).