Archimedes and Continued Fractions*

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It is to Archimedes that we owe the inequalities

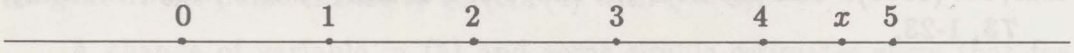
$$3\frac{10}{71} < \pi < 3\frac{1}{7} .$$

The letter π is the first letter of the Greek word for perimeter, and is understood to mean the circumference of the circle of diameter 1.

Continued fractions were invented, perhaps one should say discovered, as the most efficient way to find rational numbers close to a given number x. The most important ingredient in defining continued fractions is the "bracket function":

$$[x] \text{ is the largest integer } \leq \text{ x} .$$

It might be helpful to consider a picture.



For the x in the picture, $[x] = 4$.

From the definition of $[x]$, we see that

$$0 \leq x - [x] < 1,$$

and it is convenient to set

$$\{x\} = x - [x],$$

so that

$$x = [x] + \{x\},$$

and we have decomposed x into the sum of two numbers, $[x]$, the integral part of x, and $\{x\}$, the fractional part of x.

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Suppose $\{x\} \neq 0$, so that

$$0 < \{x\} < 1.$$

Then $\frac{1}{\{x\}} > 1$, and we can thus form the integral part $\left[\frac{1}{\{x\}}\right]$ of $\frac{1}{\{x\}}$, and we see that

$$x = [x] + \frac{1}{\left[\frac{1}{\{x\}}\right] + \left\{\frac{1}{\{x\}}\right\}} .$$

If $\left\{\frac{1}{\{x\}}\right\} \neq 0$, we can repeat the preceding construction. Perhaps it is worthwhile to take an example. Suppose $x = \frac{18}{13}$. We have

$$x = 1 + \frac{5}{13} , \qquad [x] = 1 , \qquad \{x\} = \frac{5}{13} .$$

Then

$$\frac{1}{\{x\}} = \frac{13}{5} = 2 + \frac{3}{5} , \qquad \left[\frac{1}{\{x\}}\right] = 2 , \qquad \left\{\frac{1}{\{x\}}\right\} = \frac{3}{5} ,$$

$$\frac{5}{3} = 1 + \frac{2}{3} , \qquad \frac{3}{2} = 1 + \frac{1}{2} , \qquad 2 = 2 + 0 ;$$

so

$$x = 1 + \frac{5}{13} = 1 + \frac{1}{\frac{13}{5}}$$

$$= 1 + \frac{1}{2 + \frac{3}{5}} = 1 + \frac{1}{2 + \frac{1}{\frac{5}{3}}}$$

$$= 1 + \frac{1}{2 + \frac{1}{1 + \frac{2}{3}}} = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{3}{2}}}}$$

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This suggests the following definition:

If n is an integer ≥ 1, set $[n] = n$; if $n_1, n_2, \ldots, n_r$ are integers ≥ 1, and $[n_1, \ldots, n_r]$ has been defined, set

$$[n_1, \ldots, n_r] = n_1 + \frac{1}{[n_2, \ldots, n_r]} .$$

Example.

$$[n_1, n_2] = n_1 + \frac{1}{n_2} ,$$

$$[n_1, n_2, n_3] = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3}} ,$$

$$[n_1, n_2, n_3, n_4] = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4}}} ,$$

$$[n_1, n_2, n_3, n_4, n_5] = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cfrac{1}{n_5}}}} .$$

With this definition, we get that

$$\frac{18}{13} = [1, 2, 1, 1, 2] ,$$

and we say that [1,2,1,1,2] is the continued fraction for $\frac{18}{13}$.

It is an important property of numbers that if x is any number ≥ 1, then x has a representation as a continued fraction, provided we allow continued fractions to continue indefinitely. If x is a rational number ≥ 1, then there are integers $n_1, \ldots, n_r \geq 1$ such that

$$x = [n_1, n_2, \ldots, n_r] ,$$

and these integers $n_1, \ldots, n_r$ are uniquely determined by x (as we saw in examining $\frac{18}{13}$). If x is not a rational number (i.e., if x is not a ratio of two integers), then there are integers $n_1, n_2, \ldots, n_r, \ldots$ (continuing forever) such that for each $r = 1, 2, \ldots$, $[n_1, \ldots, n_r]$ is the rational number represented by taking only the first r terms of the continued fraction expansion for x. Thus,

$$n_1 = [x], \qquad n_2 = \left[\frac{1}{\{x\}}\right], \qquad \text{etc.}$$

Let's play this game with a famous irrational number: $\sqrt{3}$. We have

$$1 < \sqrt{3} < 2,$$

and so

$$\left[\sqrt{3}\right] = 1, \quad \left\{\sqrt{3}\right\} = \sqrt{3} - 1.$$

$$\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}+1)} \cdot \frac{1}{(\sqrt{3}-1)} = \frac{\sqrt{3}+1}{2}\,.$$

$$\frac{1+\sqrt{3}}{2} = 1 + \frac{-1+\sqrt{3}}{2}\,; \quad \left[\frac{1+\sqrt{3}}{2}\right] = 1\,, \quad \left\{\frac{1+\sqrt{3}}{2}\right\} = \frac{-1+\sqrt{3}}{2}\,.$$

$$\frac{2}{-1+\sqrt{3}} = \frac{1+\sqrt{3}}{(1+\sqrt{3})} \cdot \frac{2}{(\sqrt{3}-1)} = 1+\sqrt{3}\,.$$

So

$$\sqrt{3} = 1 + (\sqrt{3} - 1)$$

$$= 1 + \cfrac{1}{\cfrac{1}{\sqrt{3}-1}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{-1+\sqrt{3}}{2}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{2}{-1+\sqrt{3}}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + (\sqrt{3} - 1)}},$$

and we see that

$$\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots] \qquad \text{(forever)}.$$

The numbers

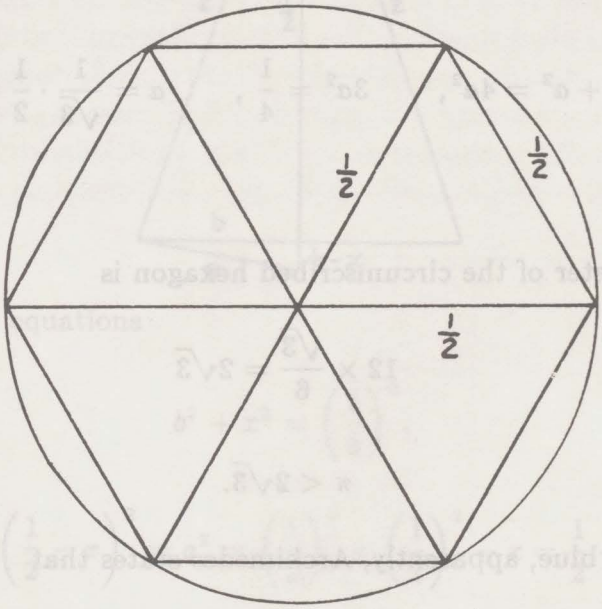
$$[1], \quad [1, 1], \quad [1, 1, 2], \quad [1, 1, 2, 1], \quad \ldots$$

are called the *convergents* to $\sqrt{3}$, and the rth number in the sequence is

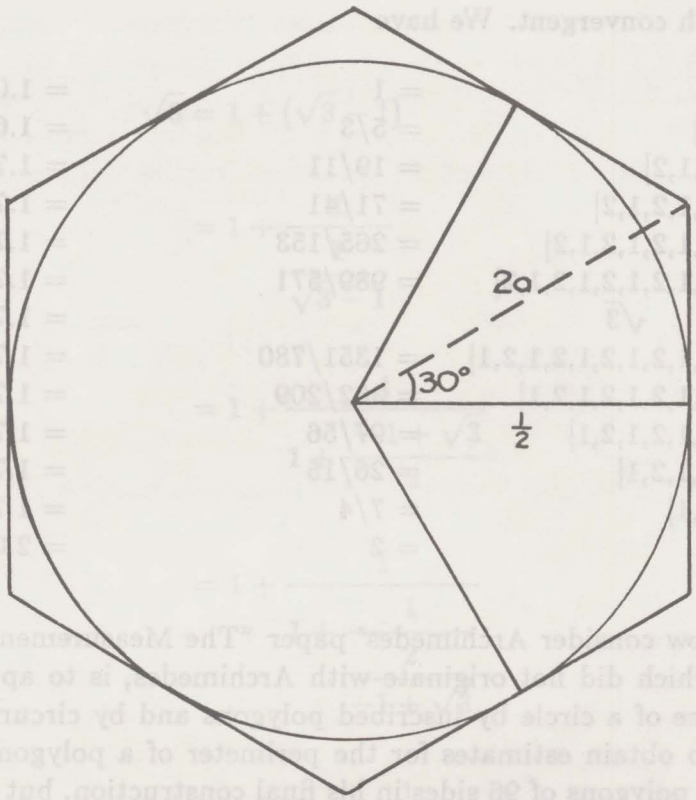
called the rth convergent. We have

[1]	$= 1$	$= 1.0000000$
[1,1,2]	$= 5/3$	$= 1.6666666$
[1,1,2,1,2]	$= 19/11$	$= 1.7272727$
[1,1,2,1,2,1,2]	$= 71/41$	$= 1.7317073$
[1,1,2,1,2,1,2,1,2]	$= 265/153$	$= 1.7320261$
[1,1,2,1,2,1,2,1,2,1,2]	$= 989/571$	$= 1.7320493$
$\sqrt{3}$		$= 1.7320508$
[1,1,2,1,2,1,2,1,2,1,2,1]	$= 1351/780$	$= 1.7320513$
[1,1,2,1,2,1,2,1,2,1]	$= 362/209$	$= 1.7320574$
[1,1,2,1,2,1,2,1]	$= 97/56$	$= 1.7321428$
[1,1,2,1,2,1]	$= 26/15$	$= 1.7333333$
[1,1,2,1]	$= 7/4$	$= 1.7500000$
[1,1]	$= 2$	$= 2.0000000$

Let's now consider Archimedes' paper "The Measurement of a Circle". The idea, which did not originate with Archimedes, is to approximate the circumference of a circle by inscribed polygons and by circumscribed polygons, and to obtain estimates for the perimeter of a polygon. Archimedes uses regular polygons of 96 sides in his final construction, but he starts with hexagons.



The inscribed hexagon has a perimeter of $6 \times \frac{1}{2} = 3$, so $3 < \pi$.



$$\left(\frac{1}{2}\right)^2 + a^2 = 4a^2, \qquad 3a^2 = \frac{1}{4}, \qquad a = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{6}.$$

The perimeter of the circumscribed hexagon is

$$12 \times \frac{\sqrt{3}}{6} = 2\sqrt{3}$$

so

$$\pi < 2\sqrt{3}.$$

Out of the blue, apparently, Archimedes states that

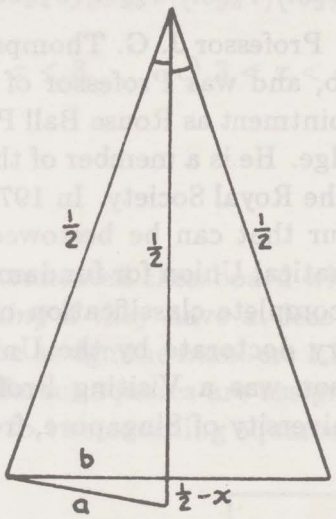
$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}.$$

Of course, it is easy to check that these inequalities are true. Simply square the terms and check that

$$\left(\frac{265}{153}\right)^2 < 3 < \left(\frac{1351}{780}\right)^2.$$

But one may ask where these particular rational numbers come from. Note that $\frac{265}{153}$ is the 9th convergent to $\sqrt{3}$ and $\frac{1351}{780}$ is the 12th convergent. I think it is safe to infer that Archimedes knew about continued fractions in some form, probably not in the same way we understand them now. For example, it would be better to take the 11th convergent rather than the 9th convergent, but even so, the fact remains that Archimedes chose *convergents*, not arbitrary rational numbers.

Given the initial approximation to $\sqrt{3}$, the remaining problem, which is an exercise in plane geometry, is to determine a when we know b.



We have the equations

$$b^2 + x^2 = \left(\frac{1}{2}\right)^2,$$

$$b^2 + \left(\frac{1}{2} - x\right)^2 = a^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - x = \frac{1}{2} - x,$$

$$a = \sqrt{\frac{1}{2} - x}, \quad x = \sqrt{\frac{1}{4} - b^2},$$

and so

$$a = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - b^2}}.$$

Archimedes' treatment of the problem is beautiful, and is worth reading over two thousand years later.

Biographical Note. Professor J. G. Thompson obtained his PhD from the University of Chicago, and was Professor of Mathematics there before assuming his current appointment as Rouse Ball Professor of Mathematics at the University of Cambridge. He is a member of the US National Academy of Sciences and a Fellow of the Royal Society. In 1970, he was awarded a Fields Medal (the highest honour that can be bestowed on a mathematician) by the International Mathematical Union for fundamental work in group theory that recently led to the complete classification of finite simple groups. He was conferred an honorary doctorate by the University of Oxford in June 1987. Professor Thompson was a Visiting Professor at the Mathematics Department, National University of Singapore, from June to August 1987.