Singapore Mathematical Olympiad Selection Test

Friday, 5 December 1986 1000 - 1300

Question Sheet

Instructions. This test consists of FORTY questions. Attempt as many questions as you can. Circle only ONE answer to each question on the Answer Sheet provided. Any question with more than one answer circled will be disallowed. There is no penalty for a wrong answer.

Each question carries an equal number of marks.

No working need to be shown on the Answer Sheet.

No calculators of any sort are allowed.

---

1. Which of the following equals $\tan\theta\sin\theta + \cos\theta$?

(a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$ (d) $\sec\theta$ (e) $\operatorname{cosec}\theta$

2. Two right circular cylinders are equal in volume. The height of the first is 4 cm while the height of the second is 3 cm. What is the radius (in cm) of the second cylinder if the first has a radius of 4 cm?

(a) $\frac{9}{4}$ (b) $\frac{8}{3}$ (c) 3 (d) $2\sqrt{3}$ (e) $\frac{8\sqrt{3}}{3}$

3. If $x = (-1)^{\frac{1}{n}}$ and n is an integer, for which value of n may x be a real number?

(a) none (b) all positive integers (c) all positive even integers
(d) all odd integers (e) all negative integers

4. Which of the following points lies on the line passing through (-1, -2) and (-3, -4)?

(a) (7,8) (b) (8,7) (c) (-8, -7) (d) (0, 1) (e) (-1, 0)

5. Let PQRS be a parallelogram. If the lengths of PQ, QR, PR and QS are a, b, c and d respectively, then $c^2 + d^2$ is equal to:

(a) $2(a^2 + b^2)$ (b) $a^2 + b^2$ (c) $\sqrt{2}(a^2 + b^2)$
(d) $4\sqrt{a^2 + b^2}$ (e) $\sqrt{3}(a^2 + b^2)$

6. Let PQRS be a square whose sides are of length 1. Two unit circles are drawn with centres at P and Q respectively. What is the area of the region bounded by the square but not by any of the two circles?

(a) $1 - \frac{4\pi}{3} + \frac{3\sqrt{3}}{4}$ (b) $1 - \frac{5\pi}{4} + \frac{3\sqrt{3}}{4}$ (c) $1 - \frac{\pi}{4}$
(d) $1 - \frac{5\pi}{4} + \frac{2\sqrt{3}}{3}$ (e) $1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}$

7. The circle inscribed in a triangle has radius r. If the sides of the triangle are a, b and c respectively and the area of the triangle is A, then r is equal to:

(a) $\frac{2A}{a+b+c}$ (b) $\frac{A}{a+b+c}$ (c) $\frac{3A}{2(a+b+c)}$ (d) $\frac{4A}{3(a+b+c)}$ (e) $A - 3(a+b+c)$

8. Let α and β be roots of the equation $5x^2 - kx + 1 = 0$, where k is positive. If $\alpha - \beta = 1$, find k.

(a) $2\sqrt{5}$ (b) $3\sqrt{5}$ (c) $2\sqrt{7}$ (d) $5\sqrt{2}$ (e) $5\sqrt{3}$

9. $11^9 - 1$ is divisible by:

(a) 1 but not 10 (b) 10 but not 100 (c) 100 but not 1000
(d) 1000 but not 10000 (e) 10000 but not 100000

10. A job can be completed by 3 men in 5 days. The same job can be completed by 4 boys in 6 days. If the job is going to be done by 5 men and 4 boys, in how many days will the job be completed?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

11. Find the number of distinct pairs (a, b) of integers satisfying the following equations:

$$\begin{cases} ab - 2b = 0 \\ a^2 + a + b = 0 \end{cases}$$

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

12. If $-1 < x < 1$, then $\big||x-1| - |x+3|\big|$ is equal to:

(a) 2 (b) $x+2$ (c) $2(x+1)$ (d) $2(1-x)$ (e) $2-x$

13. Find the number of positive common divisors of the following four numbers:

$$102, \; 408, \; 255, \; 51$$

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

14. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers such that $a_1 = 1$, $a_{n+1} = a_n + 3n$, where $(n \geq 1)$. Then a_{51} is equal to:

(a) 3571 (b) 3749 (c) 3826 (d) 3946 (e) 4016

15. If $\angle A = 3\angle B$ in $\triangle ABC$, then $\frac{BC}{AC}$ is:

(a) equal to 3 (b) less than 2 (c) greater than 3
(d) equal to $3 - 4\sin^2 B$ (e) equal to $4 - 3\cos^2 B$

16. If $f(x) = \begin{cases} 2 & \text{when } x < 1 \\ x - 1 & \text{when } x \geq 1 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{when } x < 0 \\ 1 + x & \text{when } x \geq 0 \end{cases}$, then $f(g(x))$ is equal to:

(a) $\begin{cases} 0 & \text{when } x < 0 \\ x - 1 & \text{when } x \geq 0 \end{cases}$ (b) $\begin{cases} 0 & \text{when } x < 1 \\ x - 1 & \text{when } x \geq 1 \end{cases}$

(c) $\begin{cases} 0 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases}$ (d) $\begin{cases} 0 & \text{when } x < 1 \\ x & \text{when } x \geq 1 \end{cases}$

(e) $\begin{cases} 2 & \text{when } x < 0 \\ x^2 - 1 & \text{when } x \geq 0 \end{cases}$

17. What is the largest integer value of k such that $8 - 3x^2 + x(1 + kx) = 0$ has a real root?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

18. If ABC is a triangle such that all angles are less than 90°, and $BC = 5, AC = 12, AB = x$, then:

(a) $\sqrt{119} < x < 13$ (b) $10 < x < 13$
(c) $12 < x < 13$ (d) $\sqrt{13} < x < \sqrt{119}$
(e) none of the above

19. How many arrangements of the letters

$$a,\ e,\ i,\ p,\ p,\ p,\ p,\ p$$

are there?

(a) 120 (b) 336 (c) 720 (d) 8064 (e) 40320

20. How many arrangements of the letters

$$a,\ e,\ i,\ p,\ p,\ p,\ p,\ p$$

are there, if no two vowels are adjacent?

(a) 6720 (b) 720 (c) 360 (d) 120 (e) 24

21. If a, b, c are positive integers such that $abc = 900$, find the smallest value of $a + b + c$.

(a) 29 (b) 31 (c) 32 (d) 38 (e) 40

22. Find the number of positive divisors of 13500.

(a) 18 (b) 27 (c) 36 (d) 45 (e) 48

23. In how many ways can \$1.50 be changed using any number of 5-cent, 10-cent and 20-cent coins?

(a) 60 (b) 64 (c) 68 (d) 72 (e) 76

24. Let $S_n = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots + \frac{1}{(2n-1)(2n+1)}$.

Find the value of n for which S_n is nearest to 0.48.

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

25. Find the number of solutions (x, y) of the equation $5x = 3y$ for which x, y are integers between 1 and 100 inclusive.

(a) 18 (b) 20 (c) 22 (d) 24 (e) 26

26. A circle is cut by 6 chords. Find the largest number of regions into which the circle can be divided.

(a) 12 (b) 18 (c) 20 (d) 22 (e) 32

27. Find the number of (non-congruent) triangles of sides x cm, y cm and z cm, where x, y, z are integers between 1 and 4 inclusive.

(a) 13 (b) 14 (c) 15 (d) 16 (e) 17

28. An ant is at a point A on the external rim of the base of an open cylindrical tin. A speck of sugar is inside the tin at a point B of the base diametrically opposite A. Find the shortest distance which the ant has to travel to reach the sugar if the radius of the base is 4 cm and the height of the tin is 10 cm.

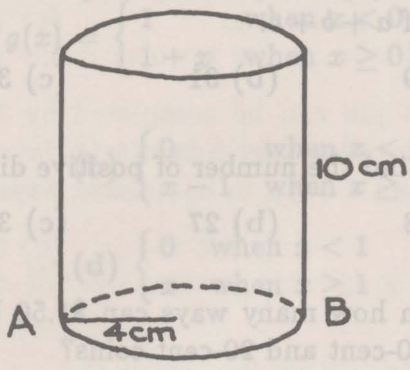
(a) $4\sqrt{\pi^2 + 25}$ cm

(b) $10 + 2\sqrt{4\pi^2 + 25}$ cm

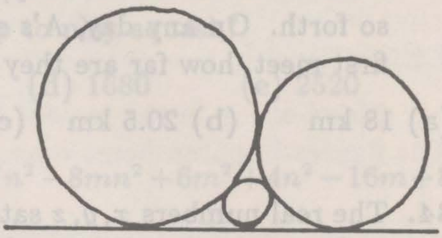
(c) $20 + 4\pi$ cm

(d) 4π cm

(e) 28 cm



29. Three circles and a line touch each other as shown below. Find the radius (in cm) of the smallest circle if the radii of the other two circles are 1 cm and 2 cm respectively.



(a) $3 - 2\sqrt{2}$

(b) $(3 - 2\sqrt{2})/2$

(c) $(3 + \sqrt{2})/10$

(d) $2(3 - 2\sqrt{2})$

(e) $(2 + \sqrt{2})/8$

30. We remove triangles from an equilateral triangle ABC as follows. At the first step, we remove the "middle triangle" $A_1B_1C_1$ where A_1, B_1, C_1 are mid-points of BC, CA, AB respectively. At the second step, we remove the "middle triangle" from each of the remaining triangles as in the first step. Repeat this procedure of removing "middle triangles". Find approximately the proportion of the area of $\triangle ABC$ that has been removed after the 5th step.

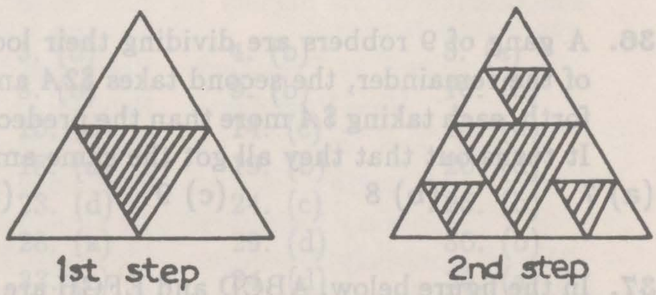
(a) 0.68

(b) 0.76

(c) 0.82

(d) 0.90

(e) 0.95



31. If k is a positive integer such that $k^n - k^3$ is divisible by 10 for all $n \geq 3$, which of the following is a possible value of k?

(a) 28 (b) 37 (c) 54 (d) 56 (e) 79

32. Find the sum of the digits of all integers from one to one million inclusive.

(a) 27,000,001 (b) 28,000,001 (c) 29,000,001 (d) 30,000,001 (e) 31,000,001

33. A and B begin to walk on the same day and in the same direction; A starts from point P, and B starts 14 km in front of A. A walks 10 km on the first day, 9 km on the second, 8 km on the third, and so forth. B walks 2 km on the first day, 4 km on the second, 6 km on the third, and so forth. On any day, A's speed is uniform, and so is B's. When they first meet, how far are they from P?

(a) 18 km (b) 20.5 km (c) 23 km (d) 29.5 km (e) 34 km

34. The real numbers x, y, z satisfy the simultaneous equations $yz = xy - z$ and $xz = zy - x$. Which of the following are possible values of z?

(a) $z \geq 0$ (b) $z \geq -1$ (c) $z \leq -4$ (d) $z \leq -5$ (e) $z \neq 0$

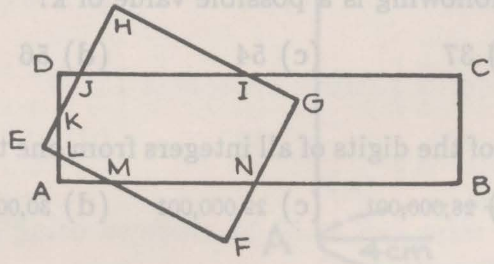
35. A piece of carpet 15 m by 4 m must be cut into N pieces and sewn together to fit a corridor that is 20 m by 3 m. What is the minimum value of N?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

36. A gang of 9 robbers are dividing their loot. The first takes $\$A$ and $\frac{1}{n}$ of the remainder, the second takes $\$2A$ and $\frac{1}{n}$ of the remainder, and so forth, each taking $\$A$ more than the predecessor plus $\frac{1}{n}$ of the remainder. It turns out that they all got the same amount. What is n?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

37. In the figure below, ABCD and EFGH are rectangles. How many circles can be drawn such that each circle passes through exactly 4 of the 14 points labelled A, B, C, ... , M, N?



(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

38. What is the maximum value of $(3-y)(y+2)(y-2)(y+1)$?

(a) -2 (b) 0 (c) 2 (d) 4 (e) 6

39. In how many ways can 8 people pair up to play squash?

(a) 105 (b) 384 (c) 840 (d) 1680 (e) 2520

40. The integers m and n are such that $3m^2n^2 - 8mn^2 + 6m^2 + 4n^2 - 16m + 8$ is negative. What is the value of m?

(a) -5 (b) -3 (c) -1 (d) 1 (e) 3

Answers.

1. (d)	2. (e)	3. (d)	4. (b)	5. (a)
6. (e)	7. (a)	8. (b)	9. (b)	10. (b)
11. (c)	12. (c)	13. (d)	14. (c)	15. (d)
16. (c)	17. (b)	18. (a)	19. (b)	20. (d)
21. (a)	22. (e)	23. (d)	24. (c)	25. (b)
26. (d)	27. (a)	28. (a)	29. (d)	30. (b)
31. (d)	32. (a)	33. (c)	34. (d)	35. (a)
36. (d)	37. (e)	38. (d)	39. (a)	40. (d)