INTERSCHOOL MATHEMATICAL COMPETITION 1986

PART A

Saturday. 28 June 1986 1000 - 1100

Attempt as many questions as you can. No calculators are allowed. Each question carries 5 marks.

1. Find the number of ways of representing 29 as a sum of 3 odd primes. (Any two representations which differ only in the order of the terms are considered to be the same.)

(a) 6 (b) 7 (c) 8 (d) 9 (e) none of the above

2. In how many ways can 2 dollars be changed into coins of denominations of 10 cents, 20 cents and 50 cents?

(a) 26 (b) 27 (c) 28 (d) 29 (e) none of the above

3. If a, b, c are positive integers such that $a + b + c = 10$, find the largest value of abc.

(a) 24 (b) 30 (c) 32 (d) 36 (e) none of the above

4. In how many ways can 100 identical candy bars be distributed among 10 boys $B_1, \ldots, B_{10}$ so that B_n gets at least n candy bars, $n = 1, \ldots, 10$?

(a) $\binom{54}{9}.10!$ (b) $\binom{45}{10}$ (c) $\binom{54}{9}$

(d) $\binom{45}{10}.10!$ (e) none of the above

[Note. $\binom{n}{r} = {}^nC_r = n!/(r!(n-r)!)$ = the number of combinations of n things taken r at a time; $n! = n(n-1) \ldots 3.2.1.$]

5. Let $p = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100}$. Then

(a) $p < 0.1$ (b) $0.1 \le p \le 0.11$ (c) $0.11 \le p < 0.12$

(d) $0.12 \le p < 0.13$ (e) $p \ge 0.13$

6. Find the "tens" digit in the decimal representation of 3^{64}.

(a) 2 (b) 4 (c) 6 (d) 8 (e) none of the above

7. The absolute value $|a|$ of a real number a is defined as follows:

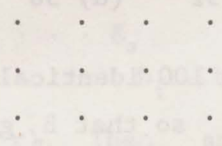
$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

Find all those values of x for which

$$\left| \frac{x - 1}{x + 1} \right| < 1.$$

(a) $x \ne -1$ (b) $x \ge 1$ (c) $0 < x < 1$ (d) $x > 0$ (e) $x > -1$

8. Find the total number of triangles whose vertices are lattice points of the following uniform lattice:



(a) 190 (b) 200 (c) 210 (d) 220 (e) none of the above

9. An insect moves along a square lattice (see Figure 1) as follows. It moves from one lattice point to an adjacent lattice point due east or due north. Find the number of paths which the insect can take in moving from O to Q so that it passes through P.

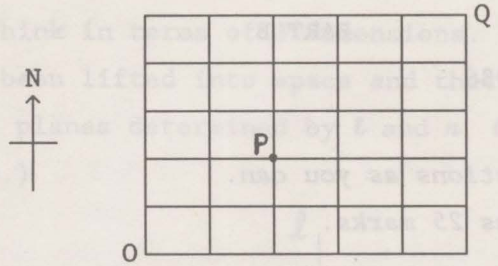


Figure 1

(a) 100 (b) 120 (c) 140 (d) 160 (e) none of the above

10. Suppose ABC is an arbitrary triangle and R, Q are arbitrary points on the sides AB, AC respectively. If P is the point of intersection of the line BC produced and the line RQ produced (see Figure 2), then the value of

$$\frac{RA}{RB} \cdot \frac{PC}{PB} \cdot \frac{QC}{QA}$$

is:

(a) 1 (b) 2 (c) 1/2 (d) 1/3 (e) none of the above.

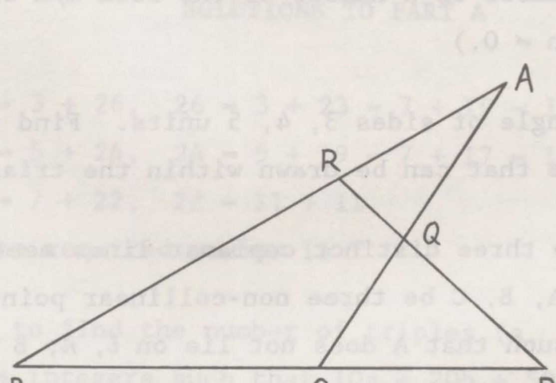


Figure 2

PART B

Saturday, 28 June 1986 1100 - 1300

Attempt as many questions as you can.

Each question carries 25 marks.

1. Let m, n be positive integers with $n \geq 3$. Show that $2^n - 1$ is not a divisor of $2^m + 1$.

2. Show that there are no positive integers a, b, c, d for which

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}.$$

3. Let $ax^2 + bx + c = 0$ be a quadratic equation with integer coefficients and $a \neq 0$. Prove that if this equation has rational solutions, then at least one of the coefficients a, b, c must be even.

(A rational number is a number of the form m/n where m, n are integers and $n \neq 0$.)

4. ABC is a triangle of sides 3, 4, 5 units. Find the area of the largest square that can be drawn within the triangle ABC.

5. Let ℓ, m, n be three distinct coplanar lines meeting at a point O, and let A, B, C be three non-collinear points in the plane of ℓ, m, n such that A does not lie on ℓ, n; B does not lie on ℓ, m; C does not lie on m, n (see Figure 1). Show how you can construct a triangle PQR using a straight edge and compasses only in such a way that P, Q, R lie on ℓ, m, n respectively, and A, B, C lie on PR, PQ, RQ respectively.

(Hint. Think in terms of 3 dimensions. Imagine that the point O has been lifted into space and that A, B, C are points on the three planes determined by ℓ and n, ℓ and m, m and n respectively.)

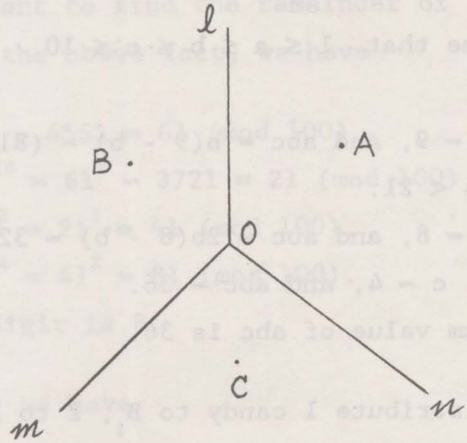


Figure 1

SOLUTIONS TO PART A

1. $29 = 3 + 26$, $26 = 3 + 23 = 7 + 19 = 13 + 13$,

   $29 = 5 + 24$, $24 = 5 + 19 = 7 + 17 = 11 + 13$.

   $29 = 7 + 22$, $22 = 11 + 11$.

   Hence the required number is 7.

2. We need to find the number of triples (a, b, c) of non-negative integers such that $10a + 20b + 50c = 200$, or $a + 2b + 5c = 20$. Since $a + 2b = 5(4 - c)$, we have $c = 0, 1, 2, 3, 4$. Consider the following cases.

   $c = 0$. $a = 2(10 - b) \Rightarrow b = 0, 1, \ldots, 10$.

   $c = 1$. $a = 15 - 2b \Rightarrow b = 0, 1, \ldots, 7$.

$c = 2.\quad a = 2(5 - b) \Rightarrow b = 0, 1, \ldots, 5.$

$c = 3.\quad a = 5 - 2b \Rightarrow b = 0, 1, 2.$

$c = 4.\quad a + 2b = 0 \Rightarrow b = 0.$

The total number of ways is $1 + 3 + 6 + 8 + 11 = 29$.

3. We may assume that $1 \le a \le b \le c \le 10$. We have the following cases.

$a = 1.\quad b + c = 9$, and $abc = b(9 - b) = (81/4) - (b - (9/2))^2 \le 81/4 < 21.$

$a = 2.\quad b + c = 8$, and $abc = 2b(8 - b) = 32 - 2(b - 4)^2 \le 32.$

$a = 3.\quad b = 3,\ c = 4$, and $abc = 36$.

Hence the maximum value of abc is 36.

4. First, we distribute 1 candy to B_1, 2 to B_2, ..., 10 to B_{10}. We then distribute the remaining 45 candies to $B_1, \ldots, B_{10}$ without any restriction. Each distribution corresponds uniquely (and conversely) to an arrangement of 45 dots and 9 x's in which the number of dots to the left of the first x denotes the number of candies given to B_1, the number of dots between the first x and the second x denotes the number of candies given to B_2, and so on. For example, ...x..xx....x... means 3 additional candies are given to B_1, 2 to B_2, none to B_3, 4 to B_4, and so on. The total number of such arrangements is $\binom{54}{9}$.

5. We have the following inequalities

$$1/2 < 2/3,\ 3/4 < 4/5,\ 5/6 < 6/7,\ \ldots,\ 99/100 < 100/101.$$

Hence $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{100}{101}$,

i.e. $p < (1/p)(1/101)$, or $p^2 = 1/101 < 1/100$, and so $p < 0.1$.

6. We use the fact that if the integers a and b have the same

remainder when divided by a positive integer m, then for any positive integer t, a^t and b^t have the same remainder when divided by m. We write this as

$$a \equiv b \pmod{m} \Rightarrow a^t \equiv b^t \pmod{m}.$$

In this case, we want to find the remainder of 3^{64} when divided by 100. Applying the above fact, we have

$$3^8 = 6561 \equiv 61 \pmod{100},$$
$$3^{16} \equiv 61^2 = 3721 \equiv 21 \pmod{100},$$
$$3^{32} \equiv 21^2 \equiv 41 \pmod{100},$$
$$3^{64} \equiv 41^2 \equiv 81 \pmod{100}.$$

Hence the "tens" digit is 8.

7. By definition, we have

$$|x - 1| = \begin{cases} x - 1 & \text{if } x \geq 1, \\ -x + 1 & \text{if } x < 1, \end{cases}$$

$$|x + 1| = \begin{cases} x + 1 & \text{if } x \geq -1, \\ -x - 1 & \text{if } x < -1. \end{cases}$$

Case 1. $x \geq 1$. The given inequality becomes $x - 1 < x + 1$, which is satisfied for all $x \geq 1$.

Case 2. $-1 \leq x < 1$. The given inequaltiy becomes $-x + 1 < x + 1$, which is satisfied for $0 < x < 1$.

Case 3. $x < -1$. The given inequality becomes $-x + 1 < -x - 1$, which is impossible.

Hence the required range is $x > 0$.

8. Any 3 non-collinear points will form a triangle. So, first we enumerate the number of ways in which we can pick 3 points which are collinear. By observation, this number is

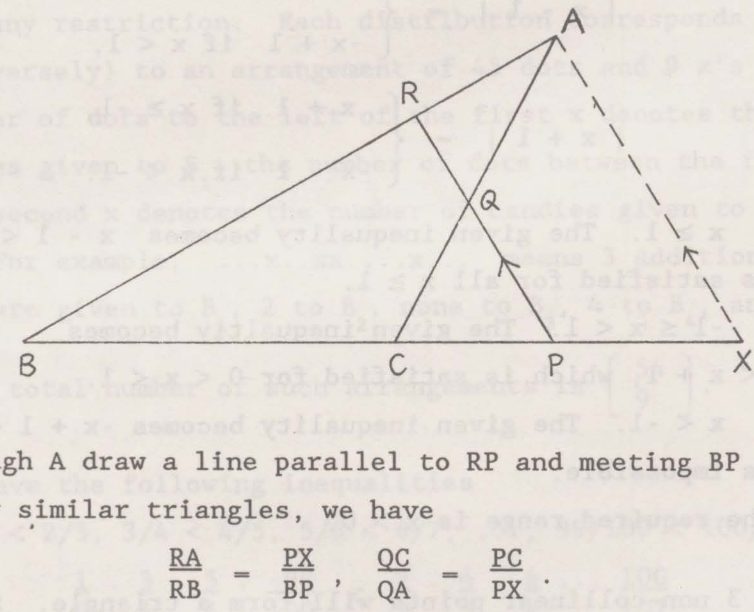
$$4 + \binom{4}{3} \cdot 3 + 4 = 20.$$

But the total number of ways of picking 3 points from the lattice is $\binom{12}{3} = 220$. Thus the required number of triangles is $220 - 20 = 200$.

9. The number of paths that the insect can take from the origin O to a point (m, n) is $\binom{m+n}{n}$ since each path can be represented as an arrangement of m E's and n N's, in which an E or an N denotes a unit step eastward or northward respectively. Hence the number of paths from O to Q and passing through P is

$$\binom{4}{2} \cdot \binom{6}{3} = 120.$$

10.



Through A draw a line parallel to RP and meeting BP produced at X. By similar triangles, we have

$$\frac{RA}{RB} = \frac{PX}{BP}, \quad \frac{QC}{QA} = \frac{PC}{PX}.$$

Hence $\frac{RA}{RB} \cdot \frac{PC}{PB} \cdot \frac{QC}{QA} = \frac{PX}{BP} \cdot \frac{PC}{PB} \cdot \frac{PC}{PX} = \left(\frac{PC}{PB}\right)^2$,

which can take infinitely many values as P moves from C to X.

SOLUTIONS TO PART B

1. Consider two cases.

Case 1. $m < n$. Then

$$2^m + 1 \le 2^{n-1} + 1 < 2^n - 1,$$

since $n \ge 3$. Hence $2^n - 1$ does not divide $2^m + 1$.

Case 2. $m \ge n$. We write $m = qn + r$ where q, r are integers with $q \ge 1$, $0 \le r < n$. Thus

$$2^m + 1 = 2^{qn}.2^r + 1 = 2^r(2^{qn} - 1) + (2^r + 1).$$

For $q \ge 1$, we have

$$2^{qn} - 1 = (2^n - 1)(2^{(q-1)n} + 2^{(q-2)n} + \ldots + 2^n + 1)$$

so that $2^n - 1$ divides $2^{qn} - 1$. Hence if $2^n - 1$ divides $2^m + 1$, then $2^n - 1$ must divide $2^r + 1$, which is impossible by Case 1 (with $r < n$).

2. Suppose there are positive integers a, b, c, d for which

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}.$$

Then $$(ad + bc)(b + d) = (a + c)bd,$$

i.e. $$ad^2 + b^2c = 0.$$

This is impossible since a, b, c, d are all positive.

3. Substituting $y = ax$ into the equation

$$ax^2 + bx + c = 0, \qquad (1)$$

we have

$$y^2 + by + ac = 0. \qquad (2)$$

Let x_1, x_2 be the roots of (1). Then $y_1 = ax_1$, $y_2 = ax_2$ are the roots of (2), and since x_1, x_2 are rational, so are y_1, y_2.

Let p/q be a rational root of (2), where p, q are coprime integers with $q > 0$. Then (2) gives

$$q^2ac = -p(p + bq),$$

and hence q divides p(p + bq). Since p, q are coprime, q divides (p + bq) and so q divides p. Hence q = 1, i.e. any rational root of (2) must be an integer.

It follows that y_1, y_2 are both integers. Moreover, we have

$$y_1 + y_2 = -b, \quad y_1 y_2 = ac.$$

Thus $abc = -y_1 y_2 (y_1 + y_2)$. Since y_1, y_2 are integers, at least one of the a, b, c must be even.

(This problem is taken from the Polish Mathematical Competition 1960.)

4.

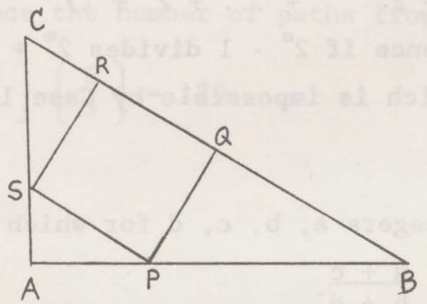


Figure (i)

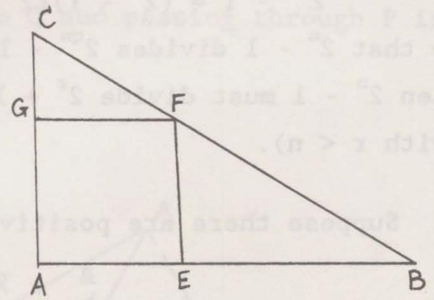


Figure (ii)

Any square $\mathcal{S}$ inside triangle ABC can be oriented so that $\mathcal{S}$ has one side on AB or BC or CA while remaining within ABC.

If one side of $\mathcal{S}$ can be made to lie on BC, then such a largest square is PQRS (Figure (i)). Let PQ = x, CR = y. Then

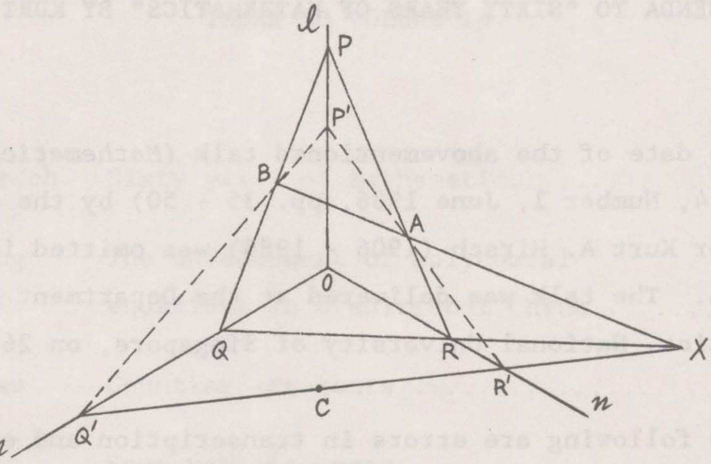
$$RS/CR = x/y = 4/3,$$

$$QB/PQ = (5 - x - y)/x = 4/3.$$

Hence $(5/x) - 1 - (3/4) = (4/3)$, i.e. $x = 60/37$.

On the other hand, if one side of $\mathcal{S}$ can be made to lie on AB or CA, then such a largest square is AEFG (Figure (ii)). Let AE = z. Then $(3 - z)/z = 3/4$, i.e. $z = 12/7$. Since $z - x > 0$, the area of the largest square is 144/49.

5.



The construction.

Step 1. Take any point P different from O on the line ℓ. Let Q and R be the points of intersection of the lines PB and m and the lines PA and n respectively. If C is on QR we are done. Otherwise proceed to the next step.

Step 2. Let X be the point of intersection of the lines AB and QR. Let Q′ and R′ be the points of intersection of XC with m and n respectively. If P′ is the point of intersection of Q′B and ℓ, then P′Q′R′ is the required triangle.

Justification.

Suppose the point O has been lifted as proposed in the hint. The objective is to find points P, Q, R on the lines ℓ, m, n respectively so that they lie in the plane $\mathscr{A}$ determined by A, B and C. In Step 1, we construct an arbitrary plane through P, A and B. If this plane contains the point C, we are done. If $\mathscr{L}$ denotes the plane determined by the lines m and n, then Q and R are points in $\mathscr{L}$. X, being collinear with Q and R, is also in $\mathscr{L}$. Since X is also in $\mathscr{A}$, XC is a line in $\mathscr{A}$. Therefore Q′ and R′ are in $\mathscr{A}$. This implies that P′ as constructed is collinear with A and R′ as required.