COUNTING ARGUMENTS*

Hirosi Nagao
Osaka University, Japan

One may have experienced in everyday life that a problem which seems to be very difficult can be solved easily by changing one's point of view.

This is also true in mathematics and flexibility is one of the most important factors in getting results in mathematics.

Notation. $|X|$ or $\#X$ denotes the number of elements of a set X, and $X^{(k)}$ denotes the family of k-subsets of X, where a k-subset means a subset of k elements.

Example 1. Let N be a set of elements. Then

$|N^{(k)}| = \binom{n}{k}$.

Count the number of elements of the set

$$X = \{ (i, S) \in N \times N^{(k)} \mid i \in S \}$$

in two ways. There are n choices of i, and for each i the number of k-subsets containing i is $\binom{n-1}{k-1}$. Hence

$$|X| = n\binom{n-1}{k-1}.$$

---

* Text of a lecture given at the prize-giving ceremony of the Singapore Mathematical Society's Interschool Mathematical Competition 1986 on 29 August 1986 in the National University of Singapore.

Professor Nagao (1925 -) is the Dean of the Faculty of Science of Osaka University, and has been a professor at Osaka University since 1968. He taught at Osaka City University and was Visiting Professor at Caltech. His research interests are in group theory and experimental designs.

On the other hand, there are $\binom{n}{k}$ choices of S, and for each S the number of $i \in S$ is k. Hence $|X| = k\binom{n}{k}$, and we have

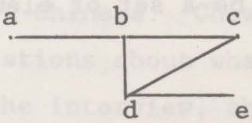
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Then by induction, we have

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ \ldots\ 1}.$$

Example 2. Let V be a set of v elements and E a subset of $V^{(2)}$. Then we call $\Gamma = (V, E)$ a *graph* and the elements of V and E are called *vertices* and *edges* respectively. For $a \in V$, the number of edges containing a is called the *degree* of a and is denoted by d(a).

For instance, the following figure



gives a graph $\Gamma = (V, E)$ with $V = \{a, b, c, d, e\}$ and

$$E = \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, e\}\},$$

and $d(a) = 1$, $d(b) = 3$, $d(c) = 2$, $d(d) = 3$, $d(e) = 1$.

Now for a graph $\Gamma = (V, E)$ count the number of elements of the set

$$X = \{(x, e) \in V \times E \mid x \in e\}$$

in two ways. Then we have

$$\sum_{x \in V} d(x) = 2|E|,$$

and the following holds.

Theorem. $\sum_{x \in V} d(x)$ is even.

As a corollary, we have the following

Corollary. (1) $\#\{\, x \in V \mid d(x)$ *is odd* $\}$ *is even.*

(2) *If* $|V|$ *is odd then there is an* $x \in V$ *with* $d(x)$ *even.*

In mathematics it happens very often that one thing assumes two or more different aspects. As an example, I will talk about the Marriage Theorem and Dilworth's Theorem.

Example 3. Let G be a set of m girls, B a set of n boys and let

$$F = \{ (g, b) \in G \times B \mid b \text{ is a boy friend of } g \}.$$

For $g \in G$ denote $\{ b \in B \mid (g, b) \in F \}$ by $B(g)$. Then the question is: under what conditions can every girl choose her husband from her boy friends? The answer is given in the following theorem.

Theorem 1. *Every girl can choose her husband from her boy friends if and only if for every* $k \le m$ *and for every* k *girls* $g_1, \ldots, g_k$,

$$\left| \bigcup_{i=1}^{k} B(g_i) \right| \le k.$$

Proof. Necessity is obvious.

Sufficiency can be proved by using Dilworth's Theorem.

Let P be a set with order $\le$, and assume that the order satisfies the following axioms:

(1) $a \le a$ for every $a \in P$,

(2) $a \le b,\ b \le a \Rightarrow a = b$ for every $a, b \in P$,

(3) $a \le b,\ b \le c \Rightarrow a \le c$ for every $a, b, c \in P$.

Then we call P a *partially ordered* set, where "partially" means that some two elements a and b may not be related with respect to $\le$ and in this case we say that a and b are *independent*. More generally, $a_1, \ldots, a_r$ are *independent* if every two elements a_i, a_j $(i \ne j)$ are independent. We denote the maximum of the number of independent elements of P by $M(P)$.

On the other hand, a subset C of P is called a *chain* if for any two elements a, b of C either $a \le b$ or else $b \le a$. If $P = C_1 \cup \dots \cup C_s$ (disjoint union) where the C_i are chains, then this union is called a *chain decomposition* of P. Let m(P) be the maximum of the number of chains appearing in a chain decomposition of P.

Theorem 2 (Dilworth). $M(P) = m(P)$.

Theorem 2 implies Theorem 1.

Proof. Let $P = G \cup B$ and define an order $\le$ on P as follows:

$$x \le y \Leftrightarrow y = x \text{ or } (y, x) \in F.$$

Then a chain is $\cdot x$ or $\begin{array}{l} g \\ | \\ b \end{array}$. Let $M(P) = m(P) = M$. Then there is a chain decomposition

$$\begin{array}{cccccc} g_1 & \cdots & g_r & g_{r+1} & \cdots & g_m \\ | & & | & & & \\ b_1 & & b_r & & & b_{r+1} \ldots b_n \end{array}$$

and $M = n + (m - r)$. We will show that $m - r = 0$.

By Theorem 2, there is an independent set

$$\begin{array}{l} g_1 \cdots g_s \\ \qquad b_1 \cdots b_t \end{array}$$

with $s + t = M = n + (m - r)$. Then

$$\{b_1, \dots, b_t\} \subseteq B - \bigcup_{i=1}^{s} B(g_i),$$

and we have $t \le n - s$. Thus $s + t = n + (m - r) \le n$ and we have $m - r = 0$.