THE DEVELOPMENT OF POLYNOMIAL EQUATIONS* IN TRADITIONAL CHINA

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When we use the term "polynomial equation", our mind conditioned with sets of mathematical notations will immediately conceptualize an equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1} + a_n = 0.$$

In the ancient world, where mathematical symbols were non-existent and mathematical expressions were in a verbalized form, how was, for instance, a quadratic equation initially conceptualized? And after the equation was formed, how was it solved? In this lecture, I shall be giving you a general survey of the development of such equations in China from antiquity to the early 14th century.

Square roots and quadratic equations

Among the existing texts on Chinese mathematics, the two earliest are the *Zhou bi suanjing* [a] (The arithmetic classic of the gnomon and the circular paths of heaven) and the *Jiu zhang suanshu* [b] (Nine chapters on the mathematical art). A conservative dating of the former would be around 100 BC while the latter is generally placed between 100 BC and 100 AD. It is well known that the *Zhou bi suanjing* contains a description of the hypotenuse diagram or *xian tu* [c] (see Fig. 1) which depicts

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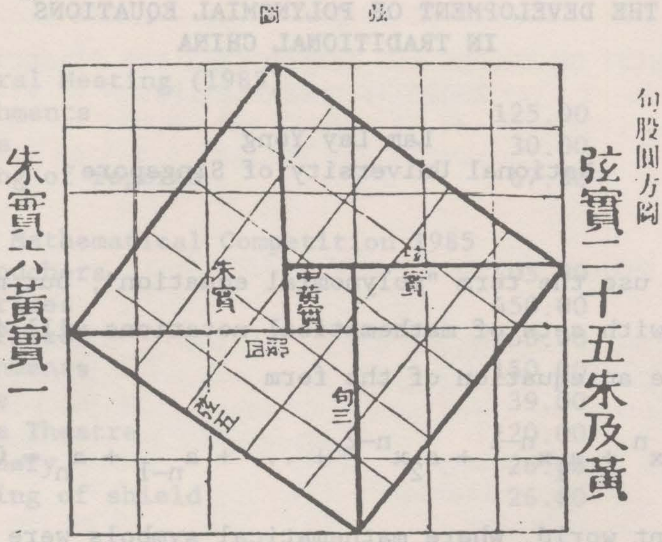


Fig. 1

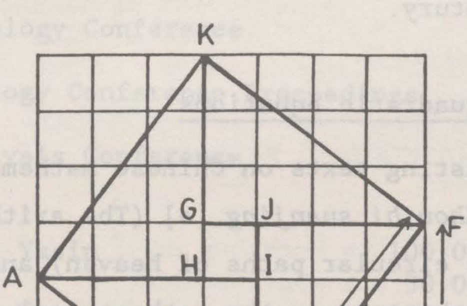


Fig. 2

one of the earliest proofs of Pythagoras theorem. However, it is not so well known that this diagram also provides one of the earliest examples in Chinese mathematics on the formulation and concept of a quadratic equation. The proof of Pythagoras theorem is generally as follows :

Referring to any one of the right-angled triangles shown in Fig. 2 (namely Δs AKH, KGF, AID, JFD, coloured in red and Δs ABD, DFE), we let the shorter orthogonal side be a, the longer orthogonal side be b and the hypotenuse be c.
Then

$$a^2 + b^2 = \text{square AHCB} + \text{square GFEC}$$
$$= \text{square AKFD}$$
$$= c^2$$

For the derivation of a quadratic equation from the diagram, b−a and c are known quantities and a is the unknown which we represent by x.
Then we have

$$\frac{1}{2}\{\text{square AKFD} - \text{square GJIH}\} = \frac{1}{2}\{c^2 - (b-a)^2\}$$
$$= 2 \text{ red triangles}$$
$$= \text{rectangle JFED}$$
$$= x(x + \overline{b-a}) .$$

Thus we have the equation

$$x^2 + (b-a)x = \frac{1}{2}\{c^2 - (b-a)^2\} .$$

The knowledge of thie equation was used to solve one of the problems (Problem 11) in the chapter on right-angled triangles in the *Jiu zhang suanshu*.

In Chapter 4 of the *Jiu zhang suanshu*, there is a general description of a method of finding the square root of a number. Like most methods in the book, this description is meant for computation by means of counting rods. The pithiness of the text can be appreciated when it is pointed out that Wang & Needham [1] took seven pages to explain this little passage consisting of one hundred and thirteen characters.

Below are some of the steps of the method illustrated by the following example.

"Find the side of a square whose area is 55225. Answer : 235."

R_5					
R_4	5	5	2	2	5
R_3					
R_2					
R_1					1

(A)

R_5			2		
R_4	5	5	2	2	5
R_3	4				
R_2	2				
R_1	1				

(B)

R_5			2		
R_4	1	5	2	2	5
R_3					
R_2	2				
R_1	1				

(C)

R_5			2		
R_4	1	5	2	2	5
R_3					
R_2		4			
R_1			1		

(D)

R_5			2	3	
R_4	1	5	2	2	5
R_3	1	2	9		
R_2		4	3		
R_1			1		

(E)

R_5			2	3	
R_4		2	3	2	5
R_3					
R_2		4	3		
R_1			1		

(F)

R_5	2 3	2 3 5
R_4	2 3 2 5	2 3 2 5
R_3		2 3 2 5
R_2	4 6	4 6 5
R_1	1	1
	(G)	(H)

In (A), 1 is placed in R_1 and 55225 in R_4.

In (B), 1 in R_1 is shifted to the left by two steps of two places each. The first component of the root, 2, is placed in the hundreds position of R_5. 2 is multiplied by 1 in R_1 and the product 2 is placed in R_2 above 1. Next, 2 is multiplied by 2 in R_2 and the product 4 is placed in R_3 as shown.

In (C), $R_4 \Rightarrow R_4 - R_3 = 55225 - 40000 = 15225$.

In (D), the number in R_2 is doubled. The numbers in R_1 and R_2 are shifted to the right by one step of two places and one place respectively.

In (E), the second component of the root, 3, is placed in the tens position of R_5. 3 is multiplied by 1 in R_1 and the product 3 is placed in R_2 above 1. Next, 3 is multiplied by 43 in R_2 and the product 129 is placed in R_3 as shown.

In (F), $R_4 \Rightarrow R_4 - R_3 = 15225 - 12900 = 2325$.

In (G), 3 in R_2 is doubled. The numbers in R_1 and R_2 are shifted to the right by one step of two places and one place respec-tively.

In (H), the third component of the root, 5, is placed in the units position of R_5. 5 is multiplied by 1 in R_1 and the product 5 is placed in R_2 above 1. Next, 5 is multiplied by 465 in R_2 and the product 2325 is placed in R_3. There is no remainder when this number is subtracted from the number in R_4. Hence the root is 235.

In this type of computation the positions of the rod numerals occupying the particular rows and columns are of vital importance. The derivation of the method is not explained in the book. This is typical of the general style of the *Jiu zhang suanshu* and also of other ancient mathematical texts where the aim is to show the reader how to solve a problem using the counting rod system. It is perhaps easier for a modern computer scientist to understand the structure of the old Chinese text than for a historian steeped in the Western tradition of deduction. A packaged software program is used by the computer scientist to obtain end-results on a computer screen. Although the underlying reasons for the steps taken are not given, it is evident that the program could only be constructed on the absolute knowledge of these reasons.

It is therefore reasonable to infer that the author(s) of the *Jiu zhang suanshu* were not only aware of the derivation of the square root method, but abstracted from it a general method suitable for computation with counting rods. There is strong evidence and it has been generally accepted that the derivation of the method for extracting a square root was originally based on a diagram as shown in Fig. 3 [2]. This is the earliest extant diagram which was given by Yang Hui [d] of the 13th century, but as early as the 3rd century, Liu Hui [e] had already given hints of such a derivation. Returning to the example on the square

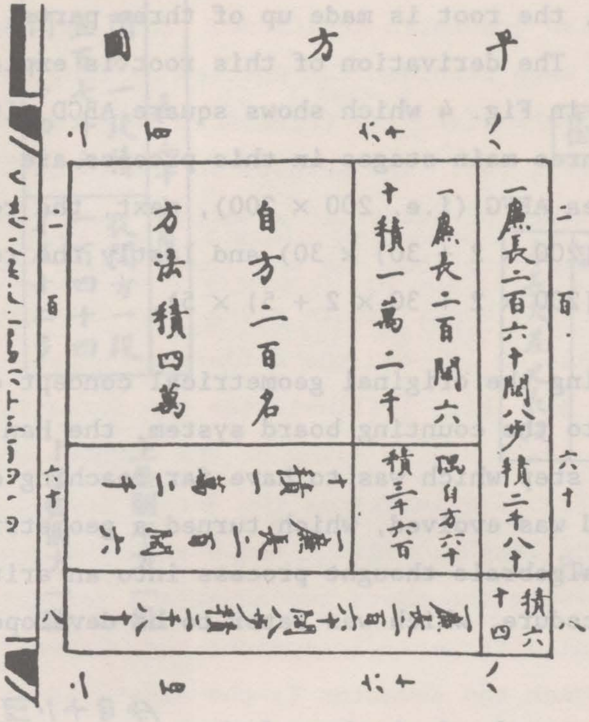


Fig. 3

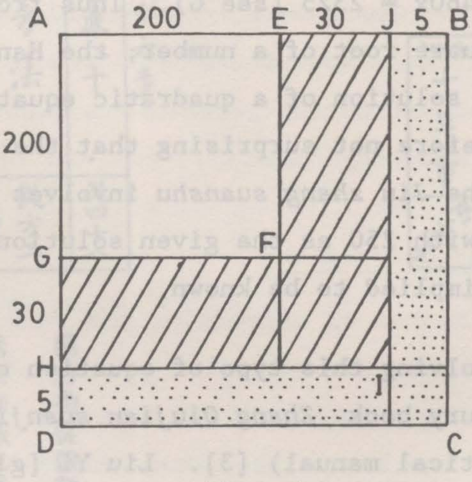


Fig. 4

root of 55225, the root is made up of three parts, viz. 200 + 30 + 5. The derivation of this root is explained geometrically in Fig. 4 which shows square ABCD with an area 55225. The three main stages in this process are : first, the removal of area AEFG (i.e. 200×200), next, the removal of area EFGHIJ (i.e. $\{200 \times 2 + 30\} \times 30$) and lastly the removal of area HIJBCD (i.e. $\{200 \times 2 + 30 \times 2 + 5\} \times 5$).

By adapting the original geometrical concept of square root extraction into the counting board system, the Han mathematicians took a unique step which was to have far reaching effects. A general method was evolved, which turned a geometrical concept involving an algebraic thought process into an arithmetic-algebraic procedure, which was later to be developed into an algorithm.

Furthermore, the depiction of the stages on the counting board showed algebraic representations of quadratic equations, which when written in our present notation are $x^2 + 400x = 15225$ (see D) and $x^2 + 460x = 2325$ (see G). Thus from the procedure of extracting the square root of a number, the Han mathematicians could also find a solution of a quadratic equation of the above type. It is therefore not surprising that the twentieth problem of Chapter 9 of the *Jiu zhang suanshu* involves the equation $x^2 + 34x = 71000$ with 250 as the given solution. The method is not given and is implied to be known.

Problems involving this type of equation can also be found in the fifth century book *Zhang Qiujian suanjing* [f] (Zhang Qiujian's mathematical manual) [3]. Liu Yi [g] (fl. 1080–1113) in his *Yi gu gen yuan* [j] (Discussions on the old sources) was responsible for writing down a most comprehensive range of

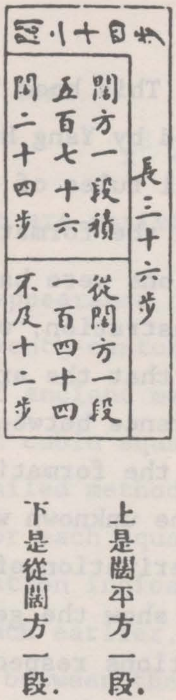


Fig. 5

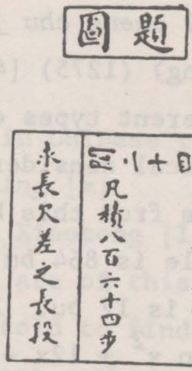


Fig. 6

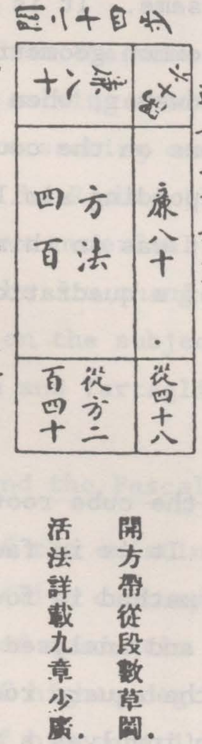


Fig. 7

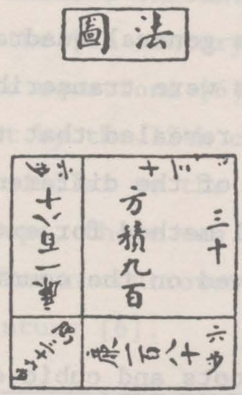


Fig. 8

methods to solve quadratic equations. This book is lost, but these methods were recorded and anlysed by Yang Hui in the *Tian mu bilei cheng chu jiefa* [i] (Practical rules of mathematics for surveying) (1275) [4]. The methods on the formation and solution of different types of quadratic equations were based on geometrical consideration. As an illustration, we take two problems from this book. It is given that the area of a rectangle is 864 *bu* [j] and the difference between its length and breadth is 12 *bu*. Fig. 5 illustrates the formation of the equation $x^2 + 12x = 864$ in terms of the unknown which is the breadth, and Fig. 6 illustrates the derivation of $x^2 - 12x = 864$ in terms of the length. Figs. 7 and 8 show the geometrical concept of solving the above two equations respectively. The diagrams reveal different dissections and shapes for different methods even when the equation is the same. It is impossible from these derivations to abstract a common geometrical method to solve a general quadratic equation. However, when these various methods were transcribed into operations on the counting board, it was revealed that there were corresponding similarities in the stages of the different operations. It was in this manner that a general method for extracting a root of a quadratic equation was perceived on the counting board.

Cube roots and cubic equations

The Chinese method of extracting the cube root of a number is similar to the square root method. It is in fact an extension of the latter. A description of this method is found in the *Jiu zhang suanshu* and has been translated and analysed by Wang and Needham [5]. Just as in the case of the square root method, a section of the cube root method, which involves a transformed cubic equation, can immediately be applied to solve a cubic equation of the form.

$$x^3 + ax^2 + bx = c ,$$

where a, b and c are positive.

The first appearance of cubic equations in Chinese texts is found in the seventh-century book *Qi gu suanjing* [k] (Continuation of ancient mathematics) by Wang Xiaotong [l]. All the twenty-eight cubic equations in this book are of this type. Although no detailed method is given, we are told to find a positive root for each equation by the cube root method. This important revelation indicates that by the seventh-century and most probably much earlier, the Chinese mathematicians were aware of a connection between the extraction of cube roots and that of cubic equations.

Working from the cube root method the Chinese encountered no difficulty in solving cubic equations. In the 9th century, the Arabs used intersecting cones to sovle cubic equations [6]. Recent studies by Rashed [7] have shown that by the 12th century, they could solve numerical cubic equations similar to the methods initiated in the *Jiu zhang suanshu*. In Europe, there was not much progress on the subject till the well-known controversy between Cardan and Tartaglia in the 16th century [6].

<u>Jia Xian [m] and the Pascal triangle</u>

A study of the existing Chinese mathematical texts shows that Jia Xian (11th century) provided the bridge which connected the solutions of quadratic and cubic equations to the solutions of equations of higher degree. Although his work is no long extant, some of his contributions were quoted by Yang Hui.

These are

1. the introduction of the Pascal triangle in connection with root extractions [8] and,
2. a distinction being made between two methods of root extraction – *li cheng shi suo* [n] (unlocking the coefficients by means of a chart) and *zeng cheng fangfa* [o] (extraction method of adding and multiplying) [9].

The Pascal triangle (Fig. 9) was constructed in connection with the extraction of roots and is associated with the first and older of the above two methods. The technique of this method involves the multiplication by 2 (for square root) and by 3 (for cube root) of certian numbers. The second method called *zeng cheng fangfa* substitutes these processes by a series of successive additions. In so doing a ladder or algorithm system is evolved and can easily be extended to extract the fourth root, to solve quartic equations or to solve polynomial equations of any degree.

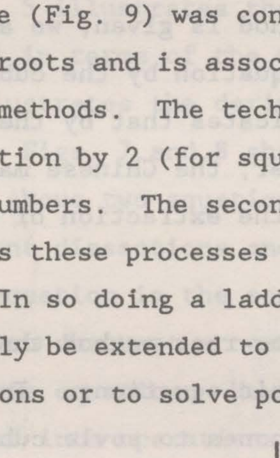


Fig. 9

The algorithm method of solving polynomial equations

What is the algorithm method? To explain this as simply as possible I have chosen a well-known problem from Qin Jiushao's [p] *Shu shu jiu zhang* [q] (Mathematical treatise in nine sections) (1247) as an illustration. This problem [10] involves the equation

$$-x^4 + 763200x^2 - 40642560000 = 0 .$$

In displaying the solution of this equation, whose root is 840, Qin had 21 panels showing the positions of the rod numerals in the various stages of calculation. The following is a very much condensed version.

	(i)	(ii)
R_6		8 0 0
R_5	− 4 0 6 4 2 5 6 0 0 0 0	− 4 0 6 4 2 5 6 0 0 0 0 0
R_4	0	0
R_3	7 6 3 2 0 0	7 6 3 2 0 0
R_2	0	0
R_1	− 1	− 1

	(iii)	(iv)
R_6	8 0 0	8 0 0
R_5	3 8 2 0 5 4 4 0 0 0 0	3 8 2 0 5 4 4 0 0 0 0 0
R_4	9 8 5 6 0 0 0 0	− 8 2 6 8 8 0 0 0 0
R_3	1 2 3 2 0 0	− 1 1 5 6 8 0 0
R_2	− 8 0 0	− 1 6 0 0
R_1	− 1	− 1

	(v)	(vi)
R_6	8 0 0	8 0 0
R_5	3 8 2 0 5 4 4 0 0 0 0	3 8 2 0 5 4 4 0 0 0 0
R_4	− 8 2 6 8 8 0 0 0 0	− 8 2 6 8 8 0 0 0 0
R_3	− 3 0 7 6 8 0 0	− 3 0 7 6 8 0 0
R_2	− 2 4 0 0	− 3 2 0 0
R_1	− 1	− 1

	(vii)	(viii)
R_6	8 4 0	8 4 0
R_5	3 8 2 0 5 4 4 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0
R_4	− 8 2 6 8 8 0 0 0 0	− 9 5 5 1 3 6 0 0 0
R_3	− 3 0 7 6 8 0 0	− 3 2 0 6 4 0 0
R_2	− 3 2 0 0	− 3 2 4 0
R_1	− 1	− 1

In (i), the coefficient of x^4 is placed in R_1, the coefficient of x^3 and x^2 in R and R respectively and the constant term in R_5.

In (ii), the number in R_1 is shifted to the left by two steps of four places each. The numbers in R_2, R_3 and R_4 are shifted to the left by two steps of three, two and one place each respectively. The first component of the root 800 is placed in R_6.

In (iii), $R_2 = R_1 \times 800 = -800$

$$R_3 = 763200 + R_2 \times 800$$

$$= 763200 + (-800 \times 800)$$

$$= 123200$$

$$R_4 = 0 + R_3 \times 800$$
$$= 123200 \times 800$$
$$= 98560000$$
$$R_5 = -40642560000 + R_4 \times 800$$
$$= -40642560000 + 98560000 \times 800$$
$$= 38205440000$$

Qin called panel (iii) the first transformation.

In (iv), $R_2 = -800 + R_1 \times 800$
$$= -800 + (-1 \times 800)$$
$$= -1600$$
$$R_3 = 123200 + R_2 \times 800$$
$$= 123200 + (-1600 \times 800)$$
$$= -1156800$$
$$R_4 = 98560000 + R_3 \times 800$$
$$= 98560000 + (-1156800 \times 800)$$
$$= -826880000$$

Panel (iv) is called the second transformation.

In (v), $R_2 = -1600 + R_1 \times 800$
$$= -1600 + (-1 \times 800)$$
$$= -2400$$
$$R_3 = -1156800 + R_2 \times 800$$
$$= -1156800 + (-2400 \times 800)$$
$$= -3076800$$

Panel (v) is called the third transformation.

In (vi), $R_2 = -2400 + R_1 \times 800$

$= -2400 + (-1 \times 800)$

$= -3200$

Panel (vi) is called the fourth transformation.

In (vii), the number in R_4 is shifted backwards to the right by one place, the number in R_3 by two places, the number in R_2 by three places and the number in R_1 by four places. The second component of the root 40 is placed in R_6.

In (viii), $R_2 = -3200 + R_1 \times 40$

$= -3200 + (-1 \times 40)$

$= -3240$

$R_3 = 3076800 + R_2 \times 40$

$= -3076800 + (-3240 \times 40)$

$= -3206400$

$R_4 = -826880000 + R_3 \times 40$

$= -826880000 + (-3206400 \times 40)$

$= -955136000$

$R_5 = 38205440000 + R_4 \times 40$

$= 38205440000 - (95513600 \times 40)$

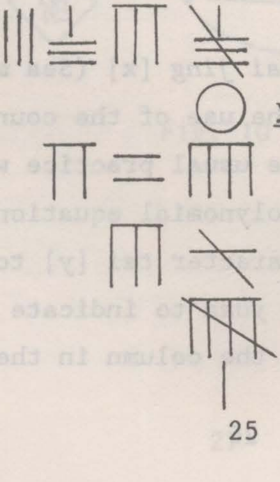
$= 0.$

Since the number in R_5 is zero, 840 in R_6 is an exact solution of the quartic equation.

By the 13th century, the mathematicians in China were familiar with the algorithm method of solving a numerical equation of any degree. Qin Jiushao, Yang Hui, Li Ye [r] and Zhu Shijie [s] used the method. This method is an algebraic one which can be applied to solve equations of any degree. It is now generally accepted that this method is similar to the techniques of solving a polynomial equation introduced by Ruffini and Horner into Europe at the beginning of the nineteenth century [11]. Both Ruffini and Horner were blissfully unaware that they were rediscovering a method which could be traced back some seventeen centuries.

Formation of polynomial equations

Accompanying the evolution of a general method of solution for a polynomial equation, changes in the formation of the equation itself were also taking place. The climax to these changes was reached with the introduction of the *tian yuan* [t] (celestial element) notation where an equation was formulated in terms of the unknown called *yuan* [u]. On a counting board, the coefficient of an unknown, x, would be placed in the row indicated by the character *yuan* and the coefficients of successive powers of x would be placed in successive rows either above or below this. For instance, the notation



represents the equation

$$x^5 - 9x^4 - 81x^3 + 729x^2 - 3888 = 0 .$$

A diagonal stroke across the last non-zero digit of the number indicates a negative term. The notation can also represent the expression $x^5 - 9x^4 - 81x^3 - 729x^2 - 3888$ and whether it represents an equation or not is judged from the meaning of the text. This notation was employed by Li Ye and Zhu Shijie in showing how a complete equation was formed from the data of a problem. One of the problems in Zhu Shijie's book *Si yuan yu jian* [v] (Jade mirror of four unknowns) written in 1303 invovles the equation

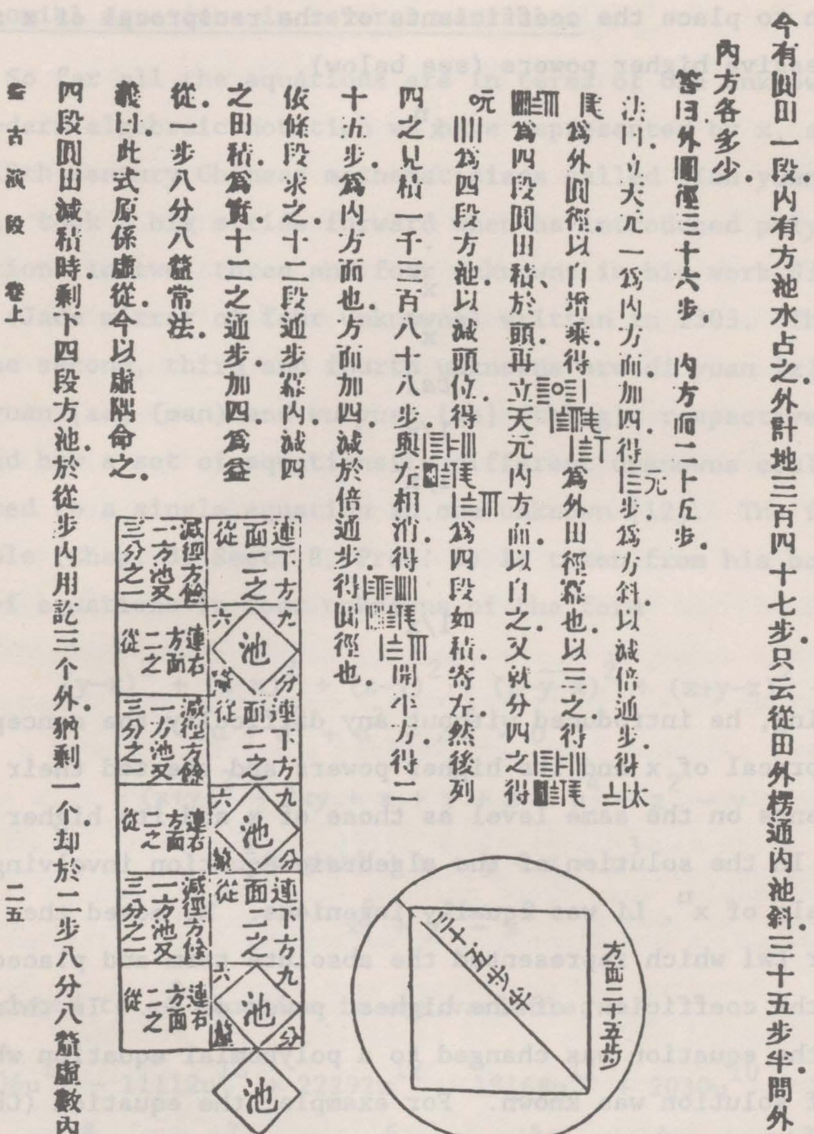
$$16x^{10} - 64x^9 + 160x^8 - 384x^7 + 512x^6 - 544x^5 + 465x^4 + 126x^3 + 3x^2 - 4x - 177162 = 0 .$$

Problem 14 (Fig. 10) of Li Ye's *Yi gu yan duan* [w] (Old mathematics in expanded sections) has the equation

$$1.88x^2 - 596.4x + 13735 = 0.$$

As with all problems in the book, the derivation of this equation by the *tian yuan* notation is shown side by side with the old geometrical method of forming the equation.

In another work *Ce yuan hai jing* [x] (Sea mirror of circle measurements), Li showed how the use of the counting board could be exploited to advantage. The usual practice was to arrange the successive coefficients of a polynomial equation in a column on the counting board with the character *tai* [y] to indicate the absolute term or the character *yuan* to indicate the coefficient of x. Li ingeniously extended the column in the opposite



第十四問

今有圓田一段內有方池水占之外計地三百四十七步只云從田外楞通內池斜三十五步半問外圓內方各多少。

答曰外圓徑三十六步。內方面二十五步。

法曰立天元一爲內方面加四得元步爲方斜以減倍通步得太爲外圓徑以自增乘得爲外田徑冪也以三之得爲四段圓田積於頭再立天元內方面以自之又就分四之得元爲四段方池以減頭位得爲四段如積寄左然後列四之見積一千三百八十八步與左相消得開平方得二十五步爲內方面也方面加四減於倍通步得爲圓徑也。

依條段求之十二段通步冪內減四之田積爲實十二之通步加四爲益從一步八分八釐常法。

義曰此式原係虛從今以虛隅命之。四段圓田減積時剩下四段方池於從步內用訖三个外猶剩一个却於二步八分八釐虛數內

益古演段　卷上　二五

Fig. 10

direction to place the coefficients of the reciprocal of x and its successive higher powers (see below).

$$
\begin{array}{c}
x^n \\
\cdot \\
\cdot \\
\cdot \\
x^2 \\
x \\
\textit{tai} \\
1/x \\
1/x^2 \\
\cdot \\
\cdot \\
\cdot \\
1/x^m
\end{array}
$$

In so doing, he introduced without any difficulty the concept of the reciprocal of x and its higher powers and treated their coefficients on the same level as those of x and its higher powers. In the solution of the algebraic equation involving the reciprocals of x^n, Li was equally ingenious. He moved the character *tai* which represented the absolute term and placed it next to the coefficient of the highest power of $\frac{1}{x}$. In this manner, the equation was changed to a polynomial equation whose method of solution was known. For example, the equation (Chap. 11, Prob. 18)

$$-x^3 - 1406x^2 - 511907x - 4730640 + 10576065600\frac{1}{x} = 0$$

becomes

$$-x^4 - 1406x^3 - 511907x^2 - 4730640x + 10576065600 = 0 .$$

Polynomial equations in several variables

So far all the equations are in terms of one unknown, which in modern algebraic notation we have represented by x, and which the 13th century Chinese mathematicians called *tian yuan*. Zhu Shijie took a big stride forward when he introduced polynomial equations in two, three and four unknowns in his work *Si yuan yu jian* (Jade mirror of four unknowns) written in 1303. The names of the second, third and fourth unknowns are *di yuan* [z] (earth) *ren yuan* [aa] (man) and *wu yuan* [ab] (thing) respectively. Zhu showed how a set of equations in different unknowns could be reduced to a single equation in one unknown [12]. The following example (Chap. 3, Sect. 8, Prob. 6) is taken from his book. A set of equations in four unknowns of the form

$$(y-x)^2 + (z-x)^2 + (z-y)^2 + (z-\overline{y-x})^2 + (x+y-z)^2 - 3u + u^2 + u^3 - 2u^4 = 0$$

$$(x+y)^2 - 2xy + x + y + z = u^4 + z^2 - y$$

$$\frac{1}{2}(x+y+z) + x + y - z = u^3$$

$$x^2 + y^2 = z^2$$

is reduced to an equation in one variable,

$$2006u^{14} - 11112u^{13} + 22292u^{12} - 19168u^{11} + 2030u^{10} + 12637u^9 - 8795u^8 - 8799u^7 + 19112u^6 - 9008u^5 - 384u^4 + 1792u^3 - 640u^2 - 768u + 1152 = 0 \quad (\text{Ans. } u = 2).$$

In so doing, Zhu established the lead by the Chinese for over four hundred and fifty years in the elimination theory of polynomial equations of several variables. In Europe, it was

Etienne Bezout who initiated the study of solving a pair of polynomial equations in two unknowns in 1764 [13].

Conclusions

The counting rod system was used for computation in China since the Warring States period (480 B.C. to 221 B.C.). The lead that the Chinese mathematicians had in the development of polynomial equations was through the use of this mode of computation. The ancient Chinese initially used geometrical models to express their algebraic notions. This tradition known as geometrical algebra was also common among the Babylonians, Greeks, Indians and Arabs. However, unlike the other civilizations, the Chinese went a step further in arithmetizing the geometric concepts on the counting board, where positions played a vital role. In taking this important step, mathematical thinking was switched from a verbalized geometric form to a rod numeral notational form. While the geometric methods were often abstruse, which make descriptions difficult and generalizations almost impossible, when these methods were transcribed on to the counting board, a new dimension in mathematical thinking became available. Patterns and symmetries were perceived on the board and these enabled methods to be generalized and extended. In such a manner, a method of finding a root of a polynomial equation of any degree to any decimal place was evolved and this was further developed to solve a set of simultaneous polynomial equations of up to four unknowns.

In the course of time, positions on the board no longer just held numerals but symbolized abstract and generalized concepts. The *tian yuan* notation was evolved from the manner in which polynomial equations were displayed and solved on a counting board. In the *tian yuan* notation, positions on the counting board imply the symbols for a polynomial equation which we use today. Thus, with the knowledge of this notation, the formulation of a polynomial equation of any degree was done as easily in 13th Century China as it is done today. The notation was flexible, as Zhu Shijie had no difficulty in extending it to express his set of polynomial equations in four unknowns. The development of polynomial equations in traditional China is an important illustration of the initial stage of how man acquired the ability to think in terms of symbols. Through this and other examples in the history of Chinese mathematics, it can be said that the concept of the symbolic form of algebra, which is an essential factor in modern mathematics, took its roots on the counting board of China.

FOOTNOTES

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3. Ang Tian Se, "A study of the Mathematical Manual of Chang Ch'iu-chien". M.A. thesis, Malaysia : University of Malaya, 1969.

4. Qian Baocong [ac], et. al., *Song Yuan shuxue shi lunwenji* [ad] (Collected essays on the history of Song and Yuan mathematics). Beijing : 1966, pp 44–47. Lam Lay Yong, *A Critical Study of the Yang Hui Suan Fa. A Thirteenth-century Chinese Mathematical Treatise.* Singapore : Singapore University Press, 1977, pp 112–125, 251–279.

5. Wang & Needham, *op. cit.*, pp 356–364.

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10. Qian Baocong, et. al., *op.cit.*, pp 45–50. Yoshio Mikami, *The Development of Mathematics in China and Japan.* New York :Chelsea Publishing Co., 1974 (2nd edition) pp 74–77. Ulrich Libbrecht, *Chinese Mathematics in the Thirteenth Century. The Shu-shu chiu-chang of Ch'in Chiu-shao.* Cambridge, Massachusetts : MIT Press, 1973, pp 180–184. Li Yan [ae] & Du Shiran [af] *Zhongguo gudai shuxue jian shi* [ag] (A short history of ancient Chinese mathematics).

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11. Wang & Needham, *op. cit.*, pp 365–381. Mikami *op. cit.*, p 77. Libbrecht, *op. cit.*, p 177–179, 189–191.

12. J. Hoe, *Les Systemes D'équations Polynômes Dans Le Siyuan Yujian (1303)*. Paris : College de France, 1977. Lam Lay Yong, "The Chinese method of solving polynomial equations of several variables". *Mathematical Medley*, 1982, *10*, 13–20.

13. Kline, *op. cit.*, p 608.

GLOSSARY

a	周髀算经	b	九章算术	c	弦图	d	杨辉
e	刘徽	f	张邱建算经	g	刘益	h	议古根源
i	田亩比类乘除捷法			j	步	k	缉古算经
l	王孝通	m	贾宪	n	立成释锁	o	增乘方法
p	秦九韶	q	数书九章	r	李冶	s	朱世杰
t	天元	u	元	v	四元玉鉴	w	益古演段
x	测圆海镜	y	太	z	地元	aa	人元
ab	物元	ac	钱宝琮	ad	宋元数学史论文集		
ae	李俨	af	杜石然	ag	中国古代数学间史		