SOLUTIONS TO PART A

1. We have $(13)^{62} = (169)\,(169^{30}) = (169)\,[(169)^2]^{15}$. The unit digit of $(169)^2$ is 1 and so the unit digit of $(169^2)^{15}$ is also 1. Hence the unit digit of $(13)^{62}$ is 9. Hence the correct answer is (d).

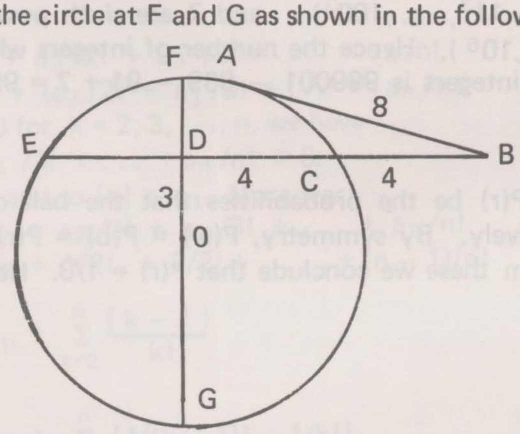
2. As $\log_{10} x + \log_{10} y = 2$, we have $\log xy = \log 100$. Hence $xy = 100$. Thus $1/(xy) = 1/100$. Now $1/x + 1/y \geqslant 2\sqrt{(1/xy)} = 1/5$. However, when $x = y = 10$, $1/x + 1/y = 1/5$. Hence the minimum value of $1/x + 1/y$ is $1/5$. The correct answer is therefore (d).

3. We have
$$f(f(x)) = [a(ax/(bx + 1))] / [b(ax/(bx + 1)) + 1]$$
$$= a^2 x/(abx + bx + 1)$$
$$= a^2 x/((ab + b)x + 1).$$
Hence if $f(f(x)) = x$, we have $ab + b = 0$ and $a^2 = 1$. From this we see that one feasible solution is $a = -1$ and b can be arbitrary. So the correct answer is (c).

4. Only the seven digits 0, 1, 2, 4, 6, 8 and 9 can be used to form positive integers less than 10,000. Thus there are 7^3 such integers ending with 1, 2, 4, 6, 8 and 9 respectively. Hence the sum of the unit digits of all these integers is equal to $7^3\,(1 + 2 + 4 + 6 + 8 + 9) = 10290$. Similarly, the sums of the tenth digits, hundredth digits, and thousandth digits of these integers are respectively equal to 10290×10, 10290×100 and 10290×1000. Hence the required sum is equal to

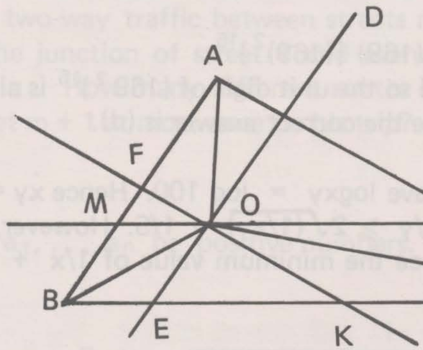
$$10290\,(1 + 10 + 100 + 1000) = 11432190.$$ The correct answer is thus (a).

5. Extent BD to intersect the circle at E and also extend OD is both directions, intersecting the circle at F and G as shown in the following figure.



We then have $(BC)(BE) = (AB)^2$ from which we get $4(DE + BD) = 64$ and so $4\,(DE + 8) = 64$. Hence $DE = 8$. Also, $(DE)(DC) = (DF)(DG)$. Thus, $8 \times 4 = (r - 3)(r + 3)$ where r is the radius of the circle, from which we get $r^2 = 41$. Hence the correct answer is (e).

6. From the figure below, we see that:



S(AOC)/S(ABC) = AF/AB since FK and AC are parallel, where S(XYZ) denotes the area of ΔXYZ. Similarly, S(AOB)/S(ABC) = BE/BC and S(COB)/S(ABC) = CN/CA. From these, it is clear that AF/AB + BE/BC + CN/CA = 1. So the correct answer is (e).

7. Let $x = \sqrt[4]{(3 + \sqrt{(3 + \sqrt{2})})}$. Then $x^4 = 3 + \sqrt{(3 + \sqrt{2})}$. Therefore $(x^4 - 3)^2 = 3 + \sqrt{2}$, i.e. $x^8 - 6x^4 + 9 = 3 + \sqrt{2}$. Hence $(x^8 - 6x^4 + 6)^2 = 2$. The correct answer is therefore (e).

8. Note that if $3 < x^3$ and $x^5 < 6$, then $3^5 < x^{15} < 6^3$. Hence we would have $243 < 216$, a contradiction. So the system has no solution for $n \geqslant 5$. For $n = 4$, $x = 1.45$ is a solution to the system. So the correct answer is (c).

9. The number of integers from 1000 to 1000000 is 999001. Among these 969 are squares of integers (namely $32^2, 33^2, \ldots, 1000^2$), 91 are cubes of integers (namely $10^3, 11^3, \ldots, 100^3$) and 7 are sixth powers of integers (namely: $4^6, 5^6, \ldots, 10^6$). Hence the number of integers which are neither squares nor cubes of integers is $999001 - 969 - 91 + 7 = 997948$. So the correct answer is (d).

10. Let P(w), P(b) and P(r) be the probabilities that the ball drawn is white, black and red respectively. By symmetry, P(w) = P(b) = P(r). Also P(w) + P(b) + P(r) = 1. From these we conclude that P(r) = 1/3. Hence the correct answer is (b).

SOLUTIONS TO PART B

1. Let A_i , B_i and C_i (i = 1, 2, 3) be the ith statements made by Grace, Helen and Mary respectively. The following combination of two statements made by each of the three ladies is consistent.: A_2 , A_3 ; B_1 , B_2 ; and C_1 , C_2 . This combination leads to the conclusion that Grace is 23, Mary 22 and Helen 25 years old. All other combinations led to a contradiction. For example, the combination: A_1 , A_2 , B_1 , B_2 , and C_1 , C_3 leads to the contradictory conclusions that Helen is both 24 and 25 years old.

2. We have : $(n!)^2 = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
$\times 1 \times 2 \times 3 \times \ldots \times (n-1) \times (n-2)$

$$= \prod_{t=0}^{n-1} (n-t) \times (t+1)$$

$$> \prod_{t=0}^{n-1} n$$

since $(n-t)(t+1) = n + nt - t2 - t = n + tn - t(t-1) > n$ for $t > 0$.
This proves that $(n!)^2 > n$, as required.

3. We first note that the equation has no solution in positive integers when x = 1 or 2. Thus we may assume that $x \geqslant 3$. Clearly, y is not divisible by 2 and 3. Hence $y = 6k \pm 1$, where $k \geqslant 1$. The given equation becomes:

$$3 \times 2^{x+1} = (6k \pm 1) = 36k^2 \pm 12k + 1,$$

i.e. $$2^{x-2} = k(3k \pm 1).$$

If k = 1, then we have (x, y) = (3, 5), (4, 7).
If $k \geqslant 2$, then $k(3k \pm 1)$ contains an odd factor while 2^{x-2} does not. We thus conclude that the only solutions in positive integers are (x, y) = (3, 5), (4, 7).

4. Let $x = a_1 + a_2/2! + a_3/3! + \ldots + a_n/n!$
$= a_1 + (a_2/2! + a_3/3! + \ldots + a_n/n!)$
Since $a_k \geqslant 0$ for $k = 2, 3, \ldots, n$, we have
$a_2/2! + a_3/3! + \ldots + a_n/n! \geqslant 0$.
Thus $x \geqslant a_1$ and so $[x] \geqslant a_1$. Moreover,

$$x = a_1 + a_2/2! + a_3/3! + \ldots + a_n/n!$$
$$\leqslant a_1 + 1/2! + 2/3! + \ldots + (n-1)/n! \qquad (a_k \leqslant k-1)$$

$$= a_1 + \sum_{k=2}^{n} \frac{(k-1)}{k!}$$

$$= a_1 + \sum_{k=2}^{n} [1/(k-1)! - 1/k!]$$

$$= a_1 + 1 - 1/n!.$$

Thus $[x] \leqslant a_1$. Since we have shown that $[x] \geqslant a_1$, we have $[x] = a_1$.

Next, we have for fixed $k \geqslant 2$.

$k!x = k!(a_1 + a_2/2! + \ldots + a_k/k! + \ldots + a_n/n!)$

$= k!(a_1 + a_2/2! + \ldots + a_k/k!) + k!\,(a_{k+1}/(k+1)! + \ldots + a_n/n!)$

$= I + II$, where

$I = k!\,(a_1 + a_2/2! + \ldots + a_k/k!)$ and $II = k!\,(a_{k+1}/(k+1)! + \ldots + a_n/n!)$.

Obviously, I is an integer. For II, we have

$$I = k!\,\Big(\sum_{i=k+1}^{n} \frac{a_i}{i!}\Big) \leqslant k!\,\Big(\sum_{i=k+1}^{n} \frac{i-1}{i!}\Big)$$

$$= k!\,\Big(\sum_{i=k+1}^{n} \frac{1}{(i-1)!} - \frac{1}{i!}\Big) = k!\,(1/k! - 1/n!)$$

$$= 1 - k!/n! < 1.$$

Hence $[k!x] = k!\,(a_1 + a_2/2! + \ldots + a_k/k!)$.

Similarly, $[(k-1)!x] = (k-1)!(a_1 + \ldots + a_{k-1}/(k-1)!)$.

Thus $[k!x] - [(k-1)!x]\,k = k!\,a_k/k! = a_k$.

5 The problem is equivalent to arranging the objects $a_1, \ldots, a_{n-1}, \ldots, b_1, \ldots, b_{m+1}$ while keeping the a's and the b's in their natural order. The answer is $(m+n)!/n-1)!(m+n)!$. However, those with $b_m\, b_{m+1}$ occuring together but not at the end are repetitions. There are $(m+n-2)!/(n-2)!m!$ of these. Therefore the answer is;

$(m+n)!\,/\,(n-1)!\,(m+1)! - (m+n-2)!/(n-2)!\,m!$.

6. We prove the identity by induction. Clearly the identity is true for $n = 1$. Assume that it is true for $n = k$. For $n = k + 1$, we have

$$\sum_{(i_1 \ldots i_{k+1})} \frac{1}{a_{i_1}\,(a_{i_1} + a_{i_2}) \ldots (a_{i_1} + \ldots + a_{i_{k+1}})}$$

$$= \frac{1}{a_1 + \ldots + a_{k+1}} \sum_{\ell=1}^{k+1} \sum_{=a_{i_{k+1}}} \frac{1}{a_{i_1}\,(a_{i_1} + a_{i_2}) \ldots (a_{i_1} + \ldots + a_{i_k})}$$

$$= 1/(a_1 + \ldots + a_{k+1})\,\Big\{\sum_{\ell=1}^{k+1} 1/\prod_{\substack{i=1\\ i\neq \ell}}^{k+1} a_i\Big\} \quad \text{(by induction hypothesis.)}$$

$$= 1/(a_1 + \ldots + a_{k+1})\,\Big\{\sum_{\ell=1}^{k+1} a_\ell/(a_1 \ldots a_{k+1})\Big\} = 1/a_1 \ldots a_{k+1}.$$

Hence the identity is true for $n = k + 1$. This proves the identity for $n \geqslant 1$.

(ii) Given that $N_1 = i_1, \ldots, N_{10} = i_{10}$, where $(i_1, \ldots i_{10})$ is a permutation of $(1, 2, \ldots, 10)$, the probability that the colours of the 10 balls drawn are all distinct is

$i_1\, i_2 \ldots i_{10}/i_1\,(i_1 + i_2) \ldots (i_1 + i_2 + \ldots + i_{10})$. But $P(N_1 = i_1, \ldots, N_{10} = i_{10}) = 1/10!$. So the probability that the colours of the 10 balls drawn are all distinct is

$$\sum_{(i_1 \ldots i_{10})} [i_1\, i_2 \ldots i_{10}/i_1\,(i_1 + i_2) \ldots (i_1 + \ldots i_{10})]\,[1/10!]$$

$= 1/10!$ by (i).