ON TRANSITIVELY ORIENTABLE GRAPHS

O. Chan, C.C. Chen and P. Slater
Department of Mathematics, National University of Singapore
Department of Mathematics and Statistics, University of Alabama in Huntsville

## 1. Introduction

A *perfect elimination ordering* [1] of a graph $G = \langle V(G), E(G) \rangle$ is an ordering $v_1, v_2, \ldots\ldots\ldots v_n$ of $V(G)$ with the property that for each i, j and k, if $i < j$, $i < k$, and $v_k, v_j \in N[v_i]$ then $v_k \in N[v_j]$, ( where $N[v]$ denotes the set of all vertices of $V(G)$ adjacent to v). Rose [1] has shown that a graph admits a perfect elimination ordering if and only if it is *chordal*, in the sense that it does not contain any cycle of length greater than three as an induced subgraph.

The concept of a transitive-orientation of a graph G as defined below bears a very close resemblance to that of a perfect elimination ordering of G:

For each graph G, we shall denote by $G^*$ a directed graph whose underlying graph is G. Let u,v be elements of $V(G)$. The uv or vu will denote the undirected edge joining u and v in G and the ordered pair (u,v) is used to denote the directed edge from u to v in $G^*$. The graph $G^*$ is said to satisfy the *transitively orientable (T-O)* condition if (x,y) is in $E(G^*)$ whenever there is a directed path from x to y in $G^*$. Equivalently, $G^*$ satisfies the (T-O) condition if (x,z) is always in $E(G^*)$ whenever (x,y) and (y,z) are in $E(G^*)$ for some y in $V(G)$. The graph G is said to be (T-O) if some $G^*$ satisfies the (T-O) condition.

The resemblance between the concepts of a perfect elimination ordering and a transitive-orientation of G can be seen from the pictorial descriptions as given in Fig. 1. In the figure, the lines joining the vertices indicate adjacency, and the wavy lines are forced by the straight lines.

The main purpose of this paper is to find a way to decide the transitive orientability of a graph.

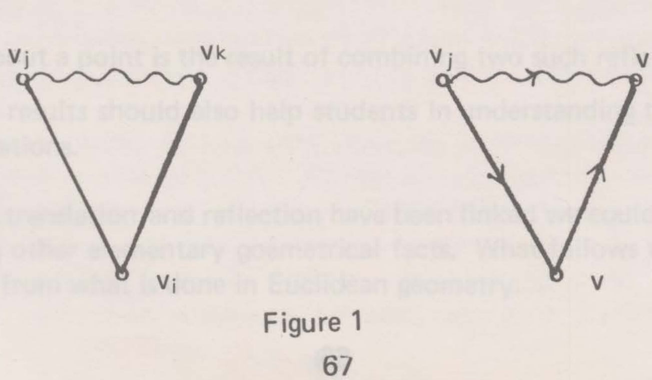


Figure 1

## 2. Factorization and Extensions of a Graph

Let G be a graph. Let T be a non-empty subset of V(G). The set T is said to be a *fair subset* of V(G) if, for each u in V(G)-T, either $N[u] \cap T = \phi$ or $N[u] \supset T$. Sometimes we simply say that T *is fair in* G to mean that T is a fair subset of V(G). Let H be a subgraph of G. Then H is called a *fair subgraph* of G if V(H) is fair in G.

Let G be a graph. Let H be a fair induced subgraph of G. The *factor graph* of G by H denoted as G/H is defined as follows:

$$V(G/H) = (V(G) - V(H)) \cup \{c\} \quad \text{where c is a new vertex not in } V(G);$$

$$E(G/H) = \{xy \in E(G) \mid x,y \in V(G)-V(H)\} \cup \{uc \mid u \in V(G)-V(H) \text{ and } N(u) \cap V(H) = \phi\}.$$

Intuitively speaking G/H is obtained form G by identifying all the vertices of H into the new vertex c. Evidently, G/H is isomorphic to some induced subgraph of G.

Next, let G and H be graphs and let a be any fixed vertex in V(G). The *extension graph of G by H via vertex a*( (denoted by G(aH)), is defined as follows:

$$V(G(aH)) = (V(G) - \{a\}) \cup V(H), \text{ (assuming that } V(G) \cap V(H) = \phi);$$

$$E(G(aH)) = E(G - a) \cup E(H) \cup \{uv \mid ua \in E(G) \text{ and } v \in V(H)\}.$$

It can easily be seen that G(aH)/H is isomorphic to G.

A vertex sequence $(v_0, v_1, \ldots, v_m)$ in G is called a *forcing sequence* if for each $i = 0, 1, \ldots m-1$, $v_i v_{i+1}$ is in E(G); whereas for each $i = 0, 1, \ldots, m-2$, $v_i v_{i+2}$ is not in E(G). (Note that $v_i$ may be the same as $v_{i+2}$). The *length* of such a sequence is defined to be m+1 (i.e. the number of terms in the sequence).

We may now define a binary relation $\sim$ on E(G) as follows:
For each ab, cd in E(G), we put

ab $\sim$ cd if and only if there exists a forcing sequence $(v_0, v_1, \ldots, v_m)$ with $v_0 = a$, $v_1 = b$, $v_{m-1} = c$ and $v_m = d$.

It is easy to see that such a relation is an equivalence relation. Moreover, for each ab in E(G), we shall denote by $S_{ab}$ the equivalence class containing ab. We say that ab is a *consistent edge* if for any xy in $S_{ab}$, the lengths of all forcing sequences of the form (a, b, . . . . x,y) are either all odd or all even.

## 3. Some Basic Lemmas

The following lemmas will be useful in the sequel:

*Lemma 1.* A graph is (T-O) iff each of its components is so.

*Proof.* Straightforward.

Hence we may, from now on, assume that all the graphs under consideration are connected, unless otherwise stated.

*Lemma 2.* All bipartite graphs are (T-O).

*Proof.* Let G be a bipartite graph where the two parts are A and B. We then turn G into a directed graph G* by putting $E(G^*) = \{(u,v) \mid uv \in E(G) \text{ and } u \in A\}$. It is then clear that G* has the (T-O) condition. □

*Corollary.* All trees are (T-O).

*Lemma 3.* A graph G is (T-O) iff all induced subgraphs of G are (T-O).

*Proof.* The sufficiency is clear as G is an induced subgraph of itself. To prove the necessity, assume that G is (T-O) and let H be any induced subgraph of G. Let G* be an orientation of G satisfying the (T-O) condition. Then H, as an induced directed subgraph of G*, provides a required orientation of H with the (T-O) condition. □

Note that G may not be (T-O) even if all of its proper induced subgraphs are (T-O). The following graph is one such example.

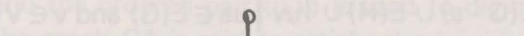

Figure 2

*Lemma 4.* Let G be a graph and $a \in V(G)$. Let H be a graph without edges. Then G is (T-O) iff G(aH) is so.

*Proof.* It is easy to see that G is an induced subgraph of G(aH). Hence G is (T-O) if G(aH) is so.

Conversely, assume that G is (T-O). Let G* be any orientation of G satisfying the (T-O) condition. We then obtain an orientation G(aH)* of G(aH) by putting:

$$E(G(aH)^*) = \{(u,v) \mid u,v \in G - a \text{ and } (u,v) \in E(G^*)\} \cup \{(u,v) \mid u \in H \text{ and } (a,v) \in E(G^*)\} \cup \{(u,v) \mid v \in H \text{ and } (u,a) \in E(G^*)\}.$$

Then it is easy to see that G(aH)* satisfies the (T-O) condition, as required. □

*Lemma 5.* Let H be a fair induced subgraph of a graph G. Then G is (T-O) if and only if G/H and H are both (T-O).

*Proof.* Assume first that G is (T-O). Then as H is given to be an induced subgraph of G and since H is fair. G/H is isomorphic to an induced of G, we see by Lemma 3 that both H and G/H are (T-O).

Conversely, assume that both H and G/H are (T-O). Then they can be orientated to yield directed graphs H* and (G/H)* respectively, both satisfying the (T-O) condition. We may then obtain an orientation G* of G by putting;

$E(G^*) = E(H^*) \cup \{ (u,v) \mid u,v \in G\text{-}H \text{ and } (u,v) \in E((G/H)) \}$
$\cup \{ (u,v) \mid u \in G\text{-}H, v \in H, \text{and } (u,a) \in E((G/H)^*) \}$
$\cup \{ (u,v) \mid u \in H, v \in G\text{-}H, \text{and } (a,v) \in E((G/H^*)) \}$,

where a is the new element in G/H not in G.

It is then easy to check that G* satisfies the (T-O) condition, as required. □

*Corollary.* Let G, G' be graphs with disjoint vertex sets. If G and G' are both (T-O), then G(aG') is (T-O) for any a in V(G).

Let S be an equivalence class of E(G) under the equivalence relation ∼ as defined before. We shall denote by ⟨S⟩ the subgraph of G as defined below:

$V(\langle S \rangle) = \{ v \in V(G) \mid v \text{ is incident with some members in } S \}$ ;
$E(\langle S \rangle) = S.$

It is clear that ⟨S⟩ is connected. Also, as the relation ∼ is an equivalence relation on E(G), we may modify the definition of consistent edges as follows:

Let S be a subset of E(G) which is an equivalence class of edges under ∼. Let ab ∈ S. It is easy to see that ab is consistent if for *some* xy in S (in fact we can just take xy = ab), the lengths of all forcing sequences of the form (a, b. . . . . . . xy) are either all odd or all even. The set S is said to be consistent if it contains at least one consistent edge. It is easy to see that if S is consistent then every edge in S is also consistent.

*Lemma 6.* Let G be a connected graph. Let S be an equivalence class of edges under ∼ . Then V(⟨S⟩) is fair.

*Proof.* If V(⟨S⟩) = V(G), then the result is true by definition. Hence suppose that V(⟨S⟩) is a proper subset of V(G). Consider any vertex u in V(G) - V(⟨S⟩) such that $N[u] \cap V(\langle S \rangle) \neq \phi$. Hence uv is in E(G) for some v in V(⟨S⟩). Let w be an element of V(⟨S⟩).Then there is a forcing sequence $(v, x_1, x_2, \ldots x_m, w)$ with $vx_1, x_i x_{i+1}, x_m w \in S$ for i = 1, 2, . . . . m-1. Then $(u,v,x_1)$ cannot be a forcing sequence since u is not in V(⟨S⟩). Hence $ux_1$ is not in E(G). Also $(u,x_1,x_2)$ cannot be a forcing sequence since u is not in V(⟨S⟩). Hence $ux_2$ is in E(G). Proceeding with the argument, we finally concluded that $(u, x_m, w)$ cannot be a forcing sequence since u is not in $V(\langle S\rangle)$. Hence uw is in E(G). Note that u and w are chosen arbitrarily. Hence $N[u] \supset V(\langle S\rangle)$ and so by definition $V(\langle S\rangle)$ is fair, completing the proof. □

*Lemma 7.* Let G be a connected graph. Let S be an equivalence class of edges under $\sim$ . Let u, v, w be elements of $V(\langle S\rangle)$ and uv, vw be elements of E(G)-S. Then $uw \in E(G)$ implies that $uw \notin S$.

*Proof.* Suppose that uw is in E(G). Partition the neighbourhood of v into two classes:

$$A_1 = \{ x \in V(G) \mid vx \in S \}$$
$$A_2 = \{ y \in V(G) \mid vy \in E(G)\text{-}S \}.$$

$A_1$ is not empty since v is in $V(\langle S\rangle)$ and is thus incident with some edge in S. $A_2$ is not empty since u, w are in $A_2$.

Note that $A_1 \cap A_2 = \phi$. For every x in $A_1$, and every y in $A_2$, (x,v,y) is not a forcing sequence, since vy is in E(G)-S and xv is in S. Hence we have established the following

*Claim.* xy is in E(G) for each x in $A_1$ and y in $A_2$.

We next prove that uw is not in S. Assume to the contrary that uw is in S. Let $x_1 \in A_1$. Then there exists a forcing sequence $(v_0, v_1, \ldots, v_{p-1}, v_p)$ from $vx_1$ to uw (since $vx_1 \sim uw$), with $v_0 = v$, $v_1 = x_1$, $v_{p-1} = u$, $v_p = w$.

Let k be the largest integer such that $v_k$ is not in $A_2$. Certainly k exists and $1 \leq k \leq p-2$. Notice that $(v_k, v_{k+1}, v)$ is not a forcing sequence as $v_{k+1}v$ is in E(G)-S. Hence $v_k v$ is in E(G). Also since $v_k$ is not in $A_2$, we must have $v_k$ in $A_1$.

Now as $v_k$ is in $A_1$ and $v_{k+2}$ is in $A_2$, $v_k v_{k+2}$ is in E(G). This contradicts the existence of $(v_k, v_{k+1}, v_{k+2})$ as a forcing sequence. Therefore the assumption that $uw \in S$ is false, completing the proof. □

*Lemma 8.* Let G be a connected graph. Let S be an equivalence class under $\sim$. Suppose $V(\langle S\rangle) = V(G)$. The G–S is disconnected.

*Proof.* If E(G) = S, the result is clear. Hence suppose that there is an edge uw in E(G)- S.

Let $M = \{ v \in V(G) \mid$ there is a path $(u_0, u_1, \ldots, u_p)$ from u to v in G with $u_0 = u$, $u_p = v$ and $u_k u_{k+1} \in E(G)\text{-}S$ for $k = 0, 1, \ldots p-1 \}$.

To prove the results, it suffices to show that M is a proper subset of V(G). To do this, we partition the neighbourhood of u into two classes as follows:

$A_1 = \{ x \in V(G) \mid ux \in S \}$,
$A_2 = \{ y \in V(G) \mid uy \in E(G)\text{-}S \}$

Then $A_1$ is non-empty since u is in $V(G) = V(\langle S \rangle)$. Also $A_2$ is non-empty as $w \in A_2$. For every x in $A_1$, and every y in $A_2$., (x,u, y) is not a forcing sequence, since uy is in E(G)-S and xu is in S. Hence xy is in E(G). By Lemma 7, we have $xy \notin E(G)\text{-}S$ as uy is in E(G)-S and xu is in S. This shows that xy is in S.

*Claim.* $M \cap A_1 = \phi$.

Indeed, let v be any element of M. Then there exists a path $(u_0, u_1, \ldots, u_p)$ from u to v with $u_0 = u$, $u_p = v$ and $u_i u_{i+1} \in E(G) - S$ for i = 0, 1, ..., p−1. Suppose to the contrary that $v \in A_1$. Then $uv \in S$. Hence $uu_{p-1} \in S$ by Lemma 7. Again $uu_{p-2} \in S$ by Lemma 7. Continuing the argument, we would eventually obtain $uu_1 \in S$, a contradiction. Hence $M \cap A_1 = \phi$, as claimed.

By the Claim, it is clear that M is a proper subset of V(G), as required. □

*Lemma 9.* If S is any equivalence class under $\sim$, then every component of E(G) - S is fair.

*Proof.* Let M be a component of E (G) − S. Let x be a vertex not in M with $N[x] \cap M \neq \phi$, say $y \in N[x] \cap M$. Then for each z in M, there exists a path $(u_0, u_1, \ldots u_p)$ in M with $u_0 = y$, $u_p = z$. Since $x \notin M$, we have $xu_0 \in S$. Then $(x, u_0, u_1)$ is not a forcing sequence. So $xu_1 \in E(G)$. By Lemma 7, $xu_1 \in S$. In like manner, we can prove that $xu_i \in S$ for i = 2, 3, ...., p. So $xz \in S$, which implies that $z \in N[x]$. Hence M is a subset of N[x], as required. □

*Corollary.* Let S and S′ be two equivalence classes of edges in a graph G under $\sim$. Then $V(\langle S \rangle) \neq V(\langle S' \rangle)$.

*Proof.* Suppose to the contrary that $V(\langle S \rangle) = V(\langle S' \rangle)$. Consider the induced subgraph H of G with vertex set $V(\langle S \rangle)$. Then $V(\langle S \rangle) = V(\langle S' \rangle) = V(H)$, $S \cup S' \subset E(H)$ and $S \cap S' = \phi$. But H-S is not disconnected since $V(\langle S' \rangle) = V(H)$. This contradicts Lemma 8. □

*Lemma 10.* Let G be a connected graph. Let S be an equivalence class of edges of G under under $\sim$. Then $\langle S \rangle$ is (T-O) if and only if S is consistent.

*Proof.* Assume that $\langle S \rangle$ is (T-O). Let $ab \in E(\langle S \rangle) = S$ and let $\langle S \rangle^*$ be an orientation of $\langle S \rangle$ satisfying the (T-O) condition. Without loss of generality, assume that $(a,b) \in E(\langle S \rangle^*)$. Take any xy in S. Suppose that (x,y) is in $E(\langle S \rangle^*)$. Consider any forcing sequence of the form (a, b, .. x, y,). This sequence must be of even length for if $(v_k, v_{k+1}, v_{k+2})$ is any segment of the above forcing sequence, then we have:

$$(v_k, v_{k+1}) \in E(\langle S \rangle^*) \Leftrightarrow (v_{k+2}, v_{k+1}) \in E(\langle S \rangle^*).$$

Hence if $(x,y)$ is an edge of $\langle S\rangle^*$, then all forcing sequence of the form $(a,b,\cdots,x,y)$ are of even length. Similarly, if $(y,x)$ is an edge of $\langle S\rangle^*$, then all forcing sequences of the form $(a,b,..,x,y)$ are of odd length. Therefore, by definition, $\langle S\rangle$ contains a consistent edge ab, and is so consistent.

Conversely, assume that S contains one consistent edge, say ab. To show that $\langle S\rangle$ is (T-O), we need to give an orientation on $\langle S\rangle$ satisfying the (T-O) condition. To do this, first put $(a,b)$ in $E(\langle S\rangle^*)$. Then for xy in S, put $(x,y)$ in $E(\langle S\rangle^*)$ if there is one forcing sequence of even length of the form $(a, b, \ldots, x. y)$ Otherwise, put $(y, x)$ in $E(\langle S\rangle^*)$. This orientation is well defined as ab is a consistent edge. With this orientation, we shall show that $\langle S\rangle^*$ satisfies the (T-O) condition.

*Claim* 1. If $(x_0, x_1), (x_1, x_2) \in E(\langle S\rangle^*)$ then $x_0x_2 \in E(\langle S\rangle)$.

Indeed, suppose to the contrary that $x_0x_2 \in E(\langle S\rangle)$. Then$(x_0, x_1, x_2)$ is a forcing sequence. By definition, there exist forcing sequences of even length of the form $\rho = (a, b, \ldots, x_0, x_1)$ and $\alpha = (a, b, \ldots, x_1, x_2)$. Now consider the forcing sequence $\rho$ derived from $\rho$ by adding just one term $x_2$ at the end, namely $\rho' = (a, b, \ldots x_0, x_1, x_2)$. $\rho'$ is a forcing sequence from ab to $x_1x_2$ of odd length and $\alpha$ is a forcing sequence from ab to $x_1x_2$ of even length. This contradicts the condition that ab is consistent. Hence $x_0x_2 \notin E(\langle S\rangle)$ is not possible, establishing Claim 1.

*Claim 2.* If $(x_0, x_1), (x_1, x_2) \in E(\langle S\rangle^*)$ then $(x_0, x_2) \in E(\langle S\rangle^*)$.

Indeed, by Claim 1 $x_0x_2 \in E(\langle S\rangle^*)$. Suppose to the contrary that $(x_0, x_2) \in E(\langle S\rangle^*)$. Then $(x_2, x_0) \in E(\langle S\rangle^*)$. As $x_0x_1 \sim x_0x_2$, there exist a forcing sequence $(v_0, v_1, \ldots, v_{p-1}, v_p)$ with $v_0 = x_0, v_1 = x_1, v_{p-1} = x_0, v_p = x_2$. As $(v_0, v_1), (v_p, v_{p-1}) \in E(\langle S\rangle^*)$, we see that p must be even. Furthermore, $(v_i, v_{i-1}), (v_i, v_{i+1}) \in E(\langle S\rangle^*)$ if $i = 2, 4, 6, \ldots, p-2$. (See Figure 3).

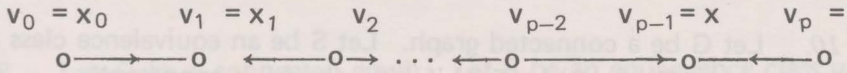


Figure 3.

We shall prove by induction on i the following proposition:

$$P(i) : (v_1, v_{p-2i}) \text{ and } (v_{p-2i-1}, v_1) \in E(\langle S\rangle^*)$$

where $i = 0,1,2, \ldots\ldots, p/2-2$.

P(i) is evidently true for $i = 0$. Asume that it is true for $i = k$ and consider $P(k+1)$. By induction hypothesis, $(v_1, v_{p-2k}), (v_{p-2k-1}, v_1) \in E(\langle S\rangle^*)$. As

$(v_{p-2k-2}, v_{p-2k-1}), (v_{p-2k-1}, v_1) \in E(\langle S \rangle^*)$, by Claim 1, $v_1 v_{p-2k-1} \in E(\langle S \rangle)$, since we have $(v_1, v_{p-2k}) \in E(\langle S \rangle^*)$. This contradicts the fact that $(v_0, v_1, \ldots, v_p)$ is a forcing sequence. Hence we must have $(v_1, v_{p-2(k+1)}) \in E(\langle S \rangle^*)$. Similarly $(v_{p-2(k+1)-1}, v_1) \in E(S)^*$ which prove the validity of $P(k+1)$. Hence $P(i)$ is true for all feasible i and in particular $(v_1, v_3) \in E(\langle S \rangle^*)$, a contradiction. This establishes Claim 2.

By Claim 2, we see that $\langle S \rangle^*$ is (T-O) as required. □

## 4. A characterization.

We are now in a position to give the following characterization for (T-O) graphs.

*Theorem 11.* Let G be a graph and S be an equivalence class of edges of G under $\sim$. Then

(i) When $E(G) = E(\langle S \rangle)$, then G is (T-O) iff S is consistent.

(ii) When $E(G) \neq E(\langle S \rangle)$ and $V(G) = V(\langle S \rangle)$, then G is (T-O) iff for any component H of G - S, both H and G/H are (T-O).

(iii) When $V(G) \neq V(\langle S \rangle)$, then G is (T-O) iff H and G/H are (T-O) where H is the induced subgraph with vertex set $V(\langle S \rangle)$.

*Proof.* (i) follows from Lemma 10.

To prove (ii), let H be any component of G-S. By Lemma 9, H is fair in G and so by Lemma 5, (ii) follows.

To proof (iii), we have by Lemma 6 that $\langle S \rangle$ and hence H is fair in G. Hence (iii) follows from Lemma 5. □

## 5. A few final remarks.

To end this paper, we wish to make the following remarks:

(1) From Theorem 11, we see from (i) that to check the transitive orientability of the graph G, we need only to decide the consistency of S. Moreover, (ii) and (iii) help us to reduce G to simpler graphs.

(2) If the graph G is (T-O), then $2 \leqslant N \leqslant 2^k$, where N is the total number of different orientations on G satisfying the (T-O) condition and k is the number of equivalence class of edges in G under $\sim$. This is clear since each equivalence class S under $\sim$ has exactly two orientations with the (T-O) condition if S is consistent and hence N is at most $2^k$. The value of N is at least 2 for each (T-O) orientation gives rise to another by reversing the direction of each arc.

(3) In [2], strongly chordal graphs are characterized by forbidden subgraphs. It will be interesting to know if (T-O) graphs can also be characterized likewise.

(4) Our arguments presented in this paper can be naturally extended to (T-O) multigraphs. It is easy to see that a multigraph is (T-O) iff its underlying simple graph is so.

References.

[1] D.J. Rose, Triangulated graphs and elimination process, J. Math. Anal, Appl. 32 (1970) 597-609.

[2] M. Farber, Characterzations of strongly chordal graphs, Discrete Maths. 43 (1983) 173-189.