Plant Spirals and Fibonacci Numbers:

A Mathematical Gold Mine

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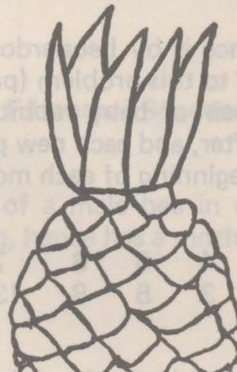
Counting the spirals on a pineapple, pine cone, or sunflower opens the door to the fascinating Fibonacci numbers and a host of mathematical concepts, with some suitable for every grade level.

Dedicated to the Memory of
ELLEN SPARER BINDMAN
1939 -- 1975



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I. Plant Spirals and the Fibonacci Sequence

On a sunflower, pineapple, pine cone, artichoke, and, in general, on the growing tip of a stem, there are conspicuous spirals, some going up to the right, some going up to the left. Count them. Here are some usual counts:

Tamarack	3, 5	Large Pineapple	13, 21
Pine Cone	5, 8	Sunflower	21, 34
Pineapple	8, 13	Giant Sunflower	34, 55

These numbers, in order of magnitude, are consecutive terms of the Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

If the n'th term is designated F_n, the sequence is generated by these conditions:

initial values: $F_1 = 1, F_2 = 1$.
recurrence relation: $F_{n+1} = F_n + F_{n-1}, n \geqslant 2$.

First mention of the sequence is by Leonardo Fibonacci of Pisa in his book *Liber Abaci* (1202) as the answer to this problem (paraphrased): A pair of rabbits, male and female, produces a pair of baby rabbits, male and female, at age 2 months and every month thereafter, and each new pair does the same. How many pairs of rabbits are there at the beginning of each month? The answer is:

month	1	2	3	4	5	6	7	8	. . .
no. of pairs	1	1	2	3	5	8	13	21	. . .

Similar sequences are obtained by taking other numbers instead of 1, 1 as the initial terms. If the first two terms are 1 and 3, the sequence is called the Lucas sequence: 1, 3, 4, 7, 11, 18, 29, . . . These terms are designated L_n, and the sequence is defined by

$$L_1 = 1, L_2 = 3; L_{n+1} = L_n + L_{n-1}, n \geqslant 2.$$

A sequence is called a generalized Fibonacci sequence if the first two terms are any two numbers, and the later terms are obtained via the recurrence relation

$$u_{n+1} = u_n + u_{n-1}, n \geqslant 2.$$

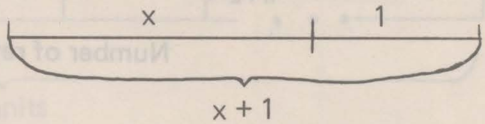
II. The Fibonacci Sequence and the Golden Section.

a) The Golden Section. If a line segment is divided into two parts so that the ratios whole segment/larger part and larger part/smaller part are equal, then the latter is called the golden section.

Let 1 = the smaller part

x = the larger part

Then x/1 = (x+1) / x, and



$$x^2 = x + 1.$$

The golden section α is the positive root of this equation.

$$\alpha = (1 + \sqrt{5})/2.$$

B) Limit of the ratio F_{n+1}/F_n. It is easily proved that this ratio approaches a limit as n approaches infinity. To identify the limit, observe that

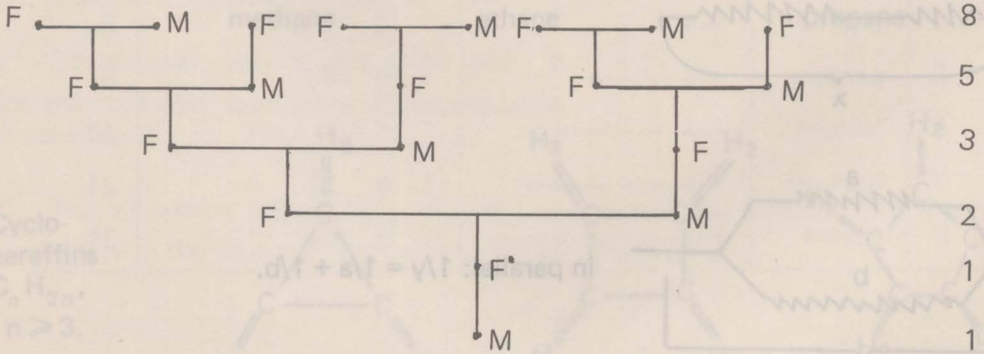
$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \cfrac{1}{\cfrac{F_n}{F_{n-1}}}$$

Let $x = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}$. Then $x = 1 + \frac{1}{x}$

Hence $x = \alpha$.

III. Other Occurrences of the Fibonacci Sequence and the Golden Section in Nature.

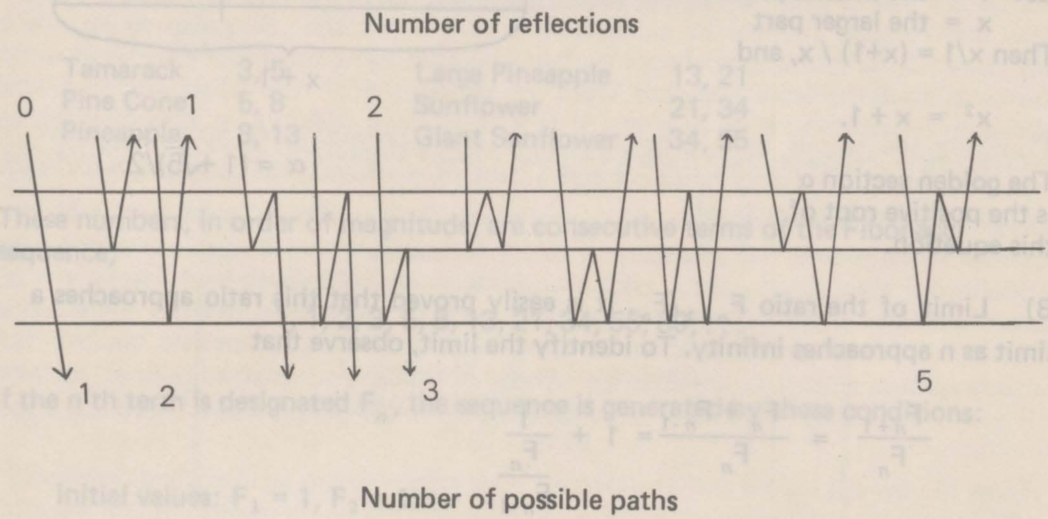
a) The number of ancestors of a male bee in each generation: (A male bee develops from an unfertilized egg, hence has a mother and no father. A female bee has two parents.)



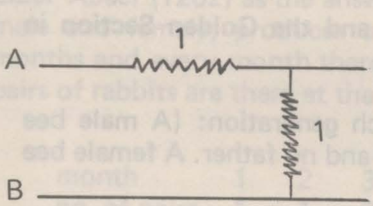
Genealogical Tree of a Male Bee

b) Reflections in Two Glass Plates

If a ray of light enters two glass plates that have a common face, and there is partial reflection at each face, and the ray emerges after n reflections, the number of possible paths is F_{n+2}.

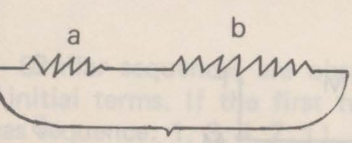


c) Ladder Network of Resistances

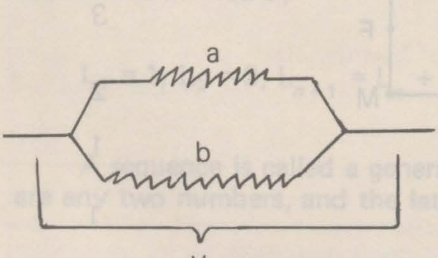


n units like this are joined to form a ladder network. If each resistance is 1 ohm, what is the total resistance r_n between A and B?

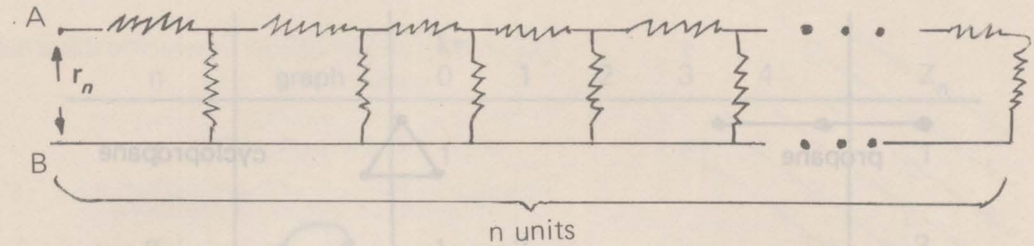
Rules for adding resistances:



in series: $x = a + b$.



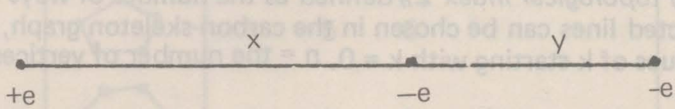
in parallel: $1/y = 1/a + 1/b$.



Prove: $r_n = F_{2n+1}/F_{2n}$.

d) Condition for Zero Potential Energy

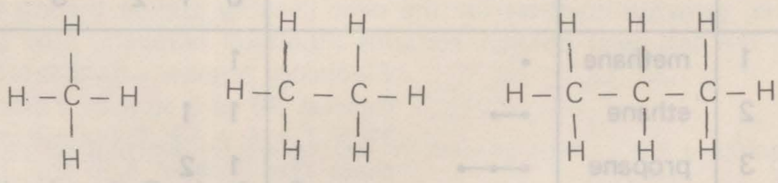
Three charges +e, −e and −e are on a line as shown below:



Find the ratio x/y if the potential energy of the system is 0.(Potential energy in a system of two charges q and q′ separated by a distance d is qq′/d.) Answer: $x/y = \alpha$.

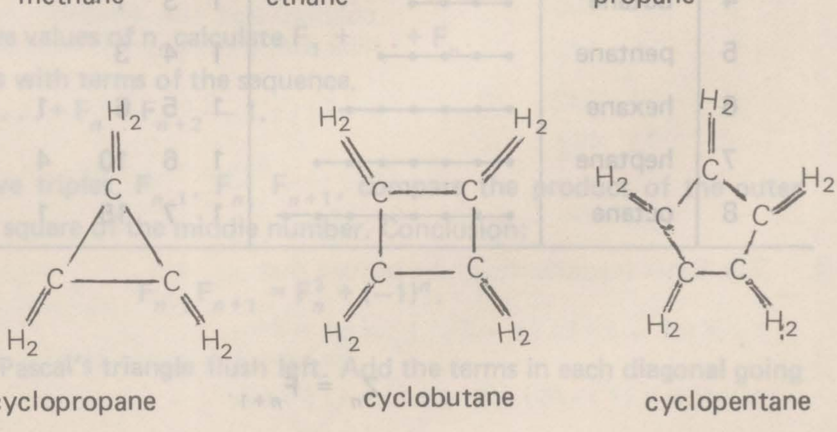
e) Topological Indices in the Paraffins

Normal Paraffins C_nH_{2n+2}, $n \geqslant 1$.



methane ethane propane

Cyclo-paraffins C_nH_{2n}, $n \geqslant 3$.



cyclopropane cyclobutane cyclopentane

The structure may be represented by the carbon skeleton. E.g.,

propane

cyclopropane

butane

cyclobutane

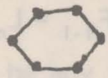
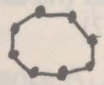
Some properties of saturated hydrocarbons (boiling point, for example) depend on the *topological index* Z_n defined as the number of ways in which a set of k disconnected lines can be chosen in the carbon-skeleton graph, summed over all possible values of k starting with k = 0. n = the number of vertices in the graph. (See below).

Normal Paraffins

n	paraffin	graph	k = 0	1	2	3	4	Z_n
1	methane		1					1
2	ethane		1	1				2
3	propane		1	2				3
4	butane		1	3	1			5
5	pentane		1	4	3			8
6	hexane		1	5	6	1		13
7	heptane		1	6	10	4		21
8	octane		1	7	15	1		34

$$Z_n = F_{n+1}.$$

Cycloparaffins

n	graph	k= 0	1	2	3	4	Z_n
1		1					1
2		1	2				3
3		1	3				4
4		1	4	2			7
5		1	5	5			11
6		1	6	9	2		18
7		1	7	14	7		29
8		1	8	20	16	2	47

$$Z_n = L_n.$$

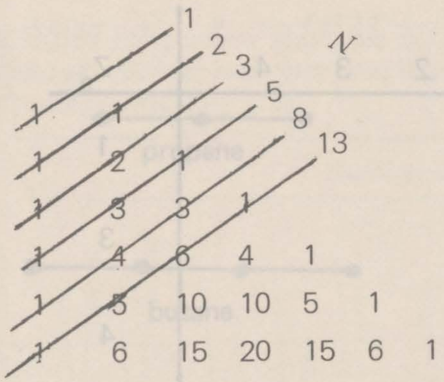
IV. Fibonacci Numbers in Grades 1 – 9

1. For successive values of n, calculate $F_1 + \ldots + F_n$.
Compare the sums with terms of the sequence.
Conclusion: $F_1 + \ldots + F_n = F_{n+2} - 1$.

2. For successive triples, F_{n-1}, F_n, F_{n+1}, compare the product of the outer numbers with the square of the middle number. Conclusion:

$$F_{n-1} F_{n+1} = F_n^2 + (-1)^n.$$

3. Write down Pascal's triangle flush left. Add the terms in each diagonal going up to the right



Let n = number of row in which a diagonal starts

k = integral part of n/2

In terms of Bernoulli numbers,

$$\binom{n-0}{0} + \binom{n-1}{1} + \ldots + \binom{n-k}{k} = F_{n+1}$$

4. Other identities easily discovered by direct computation:

$L_n^2 - 5F_n^2 = (-1)^n 4.$ $F_n L_n = F_{2n}.$

$F_n^2 + F_{n+1}^2 = F_{2n+1}$ $F_{n+1} + F_{n-1} = L_n$

5. A prediction trick. Ask your audience to start with any two numbers and generate new ones, until there are ten in all, according to the rule, "Add the last two to get the next one." Then ask the audience to add the ten numbers. While they do that, you write down your prediction of what the sum will be. It will be 11 times the seventh number. (Can you prove this rule?)

6. Fibonacci multiplication, analogous to Egyptian multiplication. Example: Multiply 19 x 34. Write 1 and 34 in two columns on two successive lines. In each column, generate numbers for the next lines by adding the last two to get the next. The left-hand column contains Fibonacci numbers. Stop when the next Fibonacci number would exceed 19. Express 19 as a sum of Fibonacci numbers. 19 = 1 + 5 + 13. Cross out all lines except those that contain 1, 5 and 13. (Keep only one of the 1's of course.) In the right-hand column add the numbers not crossed out. The sum is the answer. 19 x 34 = 646.

~~1~~	~~34~~
1	34
~~2~~	~~68~~
~~3~~	~~103~~
5	170
~~8~~	~~272~~
13	442
	646

7. The Binet formula. It can be proved that

$$F_n = (\alpha^n - \beta^n) / (\alpha - \beta)\ ;\ L_n = \alpha^n + \beta^n;$$

where $\alpha = (1 + \sqrt{5}) / 2;$ $\beta = (1 - \sqrt{5}) / 2.$

Verify by calculating F_n and L_n for some successive values of n.

Prove: $\alpha^n = (L_n + \sqrt{5} F_n) / 2.$

$\beta^n = (L_n - \sqrt{5} F_n) / 2.$

8. All the solutions in positive integers of

$x^2 - 5y^2 = +4$ or -4, are given by $x = L_n$, $y = F_n$.

In fact, if $x^2 - 5y^2 = 4$, $(x, y) = (L_2, F_2), (L_4, F_4), \ldots,$
and if $x^2 - 5y^2 = -4$, $(x, y) = (L_1, F_1), (L_3, F_3), \ldots$

This fact yields a test for determining if a positive integer N is a Fibonacci number. N is a Fibonacci number if and only if either $5y^2 + 4$ or $5y^2 - 4$ is a perfect square. Example: 219 is not a Fibonacci number because $5(219)^2 + 4 = 239809$, and $5(219)^2 - 4 = 239801$, and neither of these is a perfect square. 233 is a Fibonacci number because $5(233)^2 - 4 = 271441 = (521)^2$.

9. Let a, b, c, d be any four consecutive numbers in a generalized Fibonacci sequence. (That is, c = a+b, and d = b+c.) Prove that $(cd - ab)^2 = (ad)^2 + (2bc)^2$.

10. Solve the equation $F_{n-1} x^2 - F_n x - F_{n+1} = 0.$

11. Define the sequence $A_1, A_2, A_3, \ldots, A_n, \ldots$

by $A_1 = 2$, $A_2 = 3$; $A_n = A_{n-1} A_{n-2}$ for $n > 2$.

Find an expression for A_n.

12. Some properties that can be discovered by making appropriate calculations and observations:

a) $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$.

b) $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1.$

c) $F_3 + F_6 + F_9 + \ldots + F_{3n} = \frac{1}{2}(F_{3n+2} - 1).$

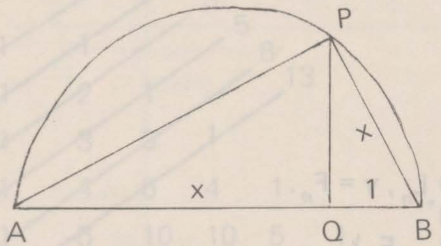
d) $F_4 + F_8 + F_{12} + \ldots F_{4n} = F^2_{2n+1} - 1.$

e) $L_n^2 - F_n^2 = 4 F_{n-1} F_{n+1}$.

d) F_n divides F_m if and only if n divides m.

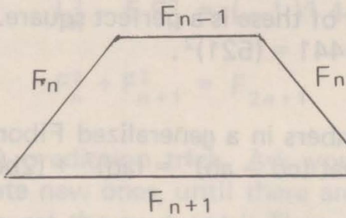
V. Fibonacci Numbers and the Golden Section in Grade 10.

1.



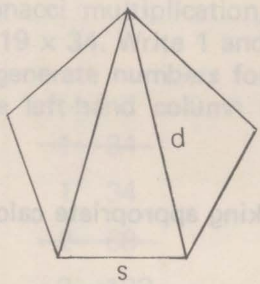
Erect a semicircle on AB as diameter. P is located on the semicircle so that the projection of AP on AB equals PB. Let QB = 1. Solve for PB. Answer: PB = x = α, since $x^2 = x + 1$.

2. Find the area of this isosceles trapezoid:



Answer: $\sqrt{3}\, F_{2n}/4$.

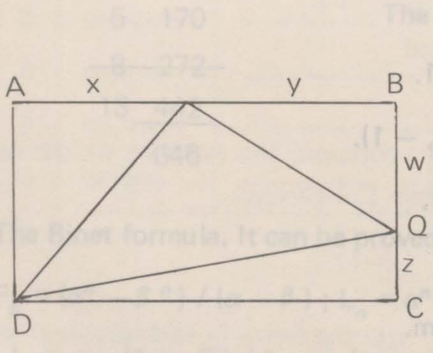
3.



In a regular pentagon,
let s = length of a side;
d = length of a diagonal.

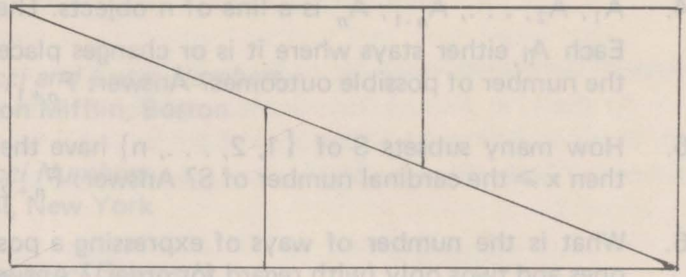
Find d/s. Answer: d/s = α.

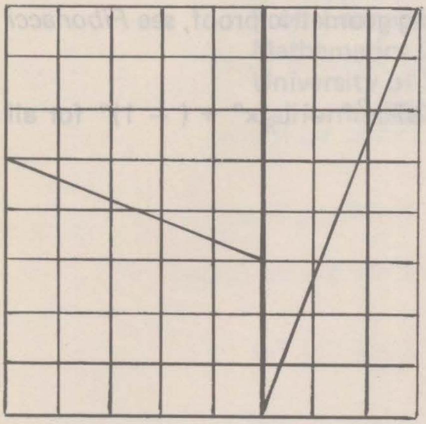
4.



In rectangle ABCD, P divides AB into segments of length x and y respectively. Q divides BC into segments of length w and z respectively. If triangles DAP, PBQ and QCD have equal areas, find the value of w/z.
Answer: w/z = α.

5. A Paradox.

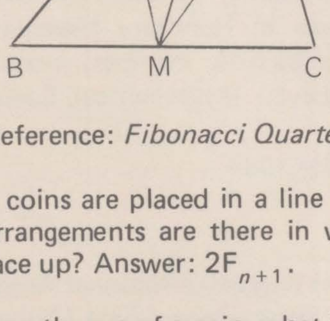




Four pieces, cut as shown from a 8 x 8 square, can be arranged to form a 5 x 13 rectangle, seeming to show that 64 = 65. Explain.

VI. Fibonacci Numbers and the Golden Section in Grades 11 – 12.

1.



Given: angle BAC = 54°.
BM is the median to BC.
AM = 1.
X is between A and B.
Y is between A and C.

What is the minimum perimeter of all possible triangles XYM?

Answer: α

Reference: *Fibonacci Quarterly*, Dec. 1974, p. 406.

2. n coins are placed in a line with either head or tail up. How many different arrangements are there in which no three consecutive coins have the same face up? Answer: $2F_{n+1}$.

3. In n throws of a coin, what is the probability that two consecutive heads will not come up?

Answer: $F_{n+1}/2^n$.

4. $A_1, A_2, \ldots, A_{n-1}, A_n$ is a line of n objects. They are permuted as follows: Each A_i, either stays where it is or changes places with a neighbour. What is the number of possible outcomes? Answer: F_{n+1}.

5. How many subsets S of $\{1, 2, \ldots, n\}$ have the property that if x is in S, then $x \geqslant$ the cardinal number of S? Answer: F_{n+2}.

6. What is the number of ways of expressing a positive integer n as a sum of ones and twos only (with regard for order)? Answer: F_{n+1}.

7. Prove that α is irrational. (For an interesting geometric proof, see *Fibonacci Quarterly*, April 1973, p. 195.)

8. Prove that $x^2 - x - 1$ is an exact divisor of $x^{2n} - L_n x^n + (-1)^n$ for all positive integers n.

9. Let Q = the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Prove: a) $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

b) $Q^2 = Q + 1$.

c) $Q + \ldots + Q^n = Q^{n+2} - Q^2$.

VII. Fibonacci Numbers and the Fabric of Mathematics

Fibonacci numbers are joined by many threads to the rest of the fabric of modern mathematics. Here is a list of some of the subjects entered if the threads are followed:

- Theory of Limits
- Linear Algebra
  - Determinants and Matrices
  - Modules and Vector Spaces
  - Spectral Theory
- Theory of Rings and Fields
- Theory of Numbers
  - Diophantine Equations
  - Congruences
  - Continued Fractions
  - Theory of Primes
- Differential Equations
- Theory of Linear Recurrence Relations
- Combinatorial Analysis
- Theory of Functions of a Complex Variable

VIII. References

V.E. Hoggatt, Jr. *Fibonacci and Lucas Numbers,* Houghton Mifflin, Boston

N.N. Vorob'ev *Fibonacci Numbers* Blaisdell, New York

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