MATHEMATICS AND APPLICATIONS

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Is mathematics useful? We who are engaged in the profession must have had occasions to wonder about this question. It is often said that mathematics is not really a science; it is an art. It is an art in the sense that in its pursuit we strive for beauty, not utility. If we study mathematics as an art, perhaps we can justify it for its own sake, although there are some people who would disparage "art for art's sake". When it comes to science, there is more reason to ask whether it is useful. Does it apply to the practical needs of our daily life? Does it contribute to the general well-being of society and mankind?

I should like to begin with a digression about art. Apparently soon after human beings learned to survive against great odds, they began already doing things which could not be considered strictly useful. For instance, there are recent discoveries of cave drawings, vivid and elaborate, of animals with which they were sharing the earth. It is not easy to see how such activities could have served their practical needs, unless they be psychic-therapeutic, in which case they were certainly unconscious or subconscious. These arts were pursued at a time when human life must have been very hard. People had to scavenge and hunt the animals for food, and avoid being eaten themselves. Yet they found the time to make drawings and sculptures. For what?

To get closer to mathematics, astronomy was developed in many ancient cultures. Of course basic knowledge of the sun and moon and some stars was necessary for planting and seafaring, but often astronomy was developed far beyond rudimentary observations. A few years ago my wife and I visited the Mayan ruins in Yucatan (Mexico) and Gautemala. You know the Mayan race was gifted in mathematics; they invented "zero". We saw sites of their religious worship, where extremely accurate measurements were made so that at a particular time of the year, a ray of sunlight would fall through some intricate design upon a precise spot. Was this feat really necessary for any application, or was it done simply to achieve perfection? Perhaps the high priests who performed these arts and sciences used these kinds of miracles

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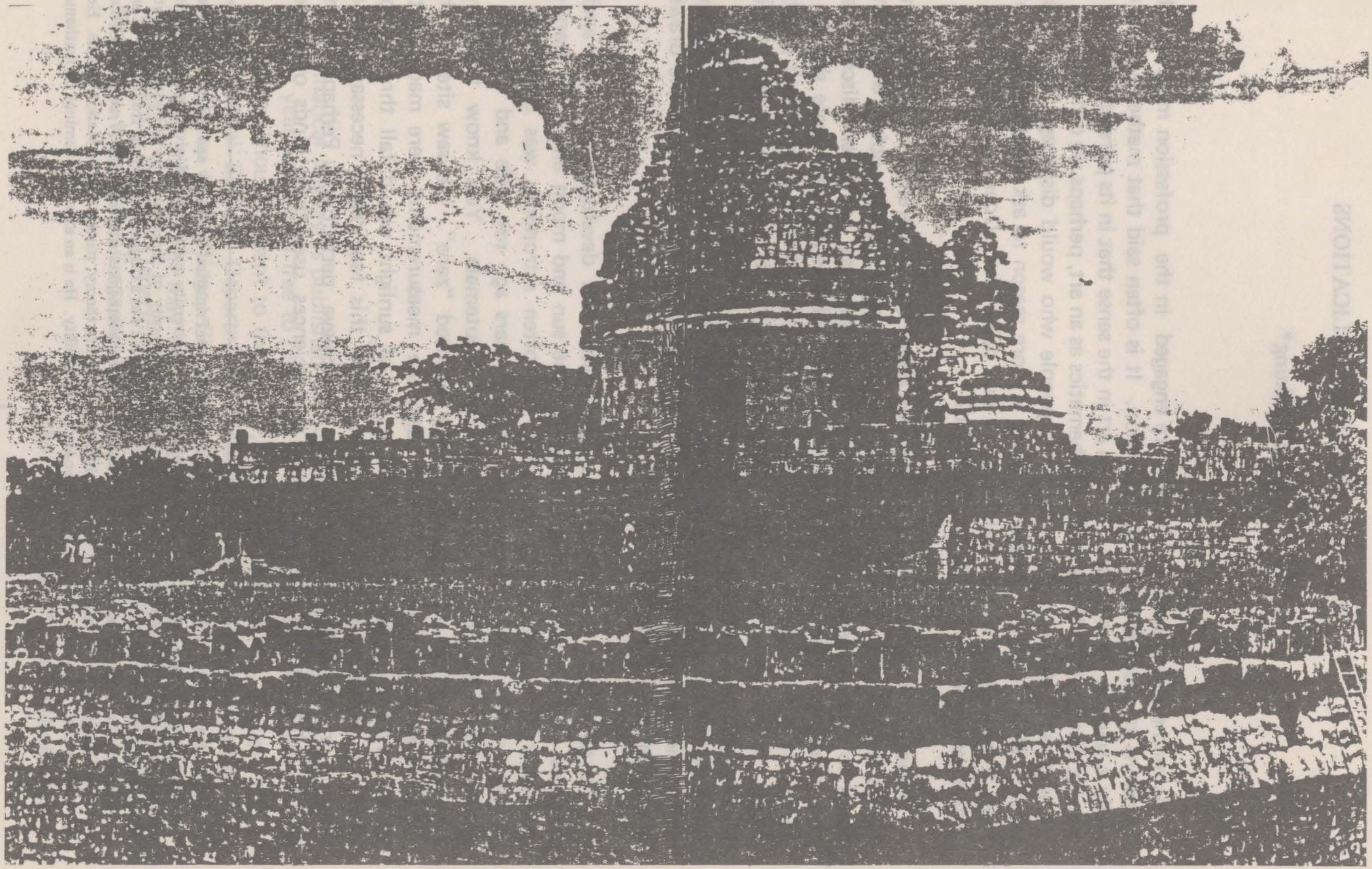

Mayan observatory (Caracol) seen from the window of our hotel.

to show off their learning and power, their affinity to gods, in order to govern their people. It is said that geometry was invented in Eygpt to keep track of the Nile-flooded land. But nobody will believe that all thirteen books of Euclid's Elements, compiled ca. 300 B.C., were needed for whatever purposes the ancient Eygptians had in mind. As another example, the celebrated Chinese Remainder Theorem was couched, as I recalled, in terms of some mundane problem. But can anyone seriously argue that even the most elementary Diophantine equations were applicable in those times, or indeed our time?

In our time we begin the study of mathematics in elementary school. When I was in middle school in China (half a century ago, alas), the three fundamental subjects were: Chinese, English and Mathematics. Now in the United States we are supposed to return to the three R's : reading, writing and arithmetic. (We are woefully inadequate in foreign languages.) Thus in the education of youths in both cultures, mathematics is a basic ingredient. Why? The traditional answer is that mathematics is prerequisite to engineering and sciences. Actually among the sciences only physics required some higher mathematics, chemistry rather little, not to mention the rest, so far as real applications are concerned. Be that as it may, everybody can agree that engineering and some scientific inventions are necessary for contemporary living. We can see these objects all around us, and we recognise the role of mathematics in their production. Thus we study calculus in the first year of college, if not already in high school. In mechanics we encounter certain notions such as velocity, acceleration and center of gravity which are treated by calculus. A little later we learn how to solve some simple differential equations, and to do some computations based on numerical and power series. The utility of mathematics to this extent is clear. I still remember how pleasantly surprised I was when told that the important notion of marginal utility in economics is a matter of differentiation. The question is how much mathematics is really needed for these and similar applications? This question intrigued me so much during the epoch when men went to the moon (an event scarcely dreamed of in my youth), that I used to quiz my acquaintances who are truly applied scientists. It might be indiscrete for me to drop a few names here, but the honest answer seems to be: very little indeed. Here I must give an example to illustrate what I mean by "really useful", on which the answer hinges. Consider the following power series expansions:

(1) $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \dots, \quad |x| < 1$$

This formula must be in all calculus textbooks, and most of us will agree that it is a useful one for all sorts of estimations and computations. (For me as a teacher of somewhat more advanced courses such as probability theory, it is a sad comment on the state of education the U.S. today that many undergraduates who have taken calculus do not know such formulas by heart.) But now let us put x=1 in (1) and obtain the numerical identity:

(2) $$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \dots,$$

The beauty of this formula should not fail to impress the fresh young mind, and the effect may be further enhanced by a companion:

$$(3) \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots,$$

which can be obtained in a similar manner from the power series for arctan. 1/

But my point is this: since the power series (1) converges for $|x| < 1$, as indicated there, can we be sure that (2) is correct? Does it not require a proof? Now if you have a good or even cheap calculator, you can compute both members of (2) and check their agreement to a certain decimal point in a matter of seconds. Does that not constitute "preponderant evidence" for the truth of (2), given its a priori plausibility from (1)? But as we in the profession know, the deduction of (2)

Zhu Chong Zhi - a Chinese mathematician of the 5th Century who gave the value of π as 355/133 or 3.1415929203

1/ I asked Professor Weil whether Euler had a proof of (2). Here is part of his answer: "Euler held that all "reasonable" processes, applied to given series, must always give the same value. (He did not explain what is 'reasonable', relying on instinct and experience to tell him that.) Thus he would not have hesitated to put x = 1 in (1)." I venture to make the point here that what Euler did not need to know cannot be useful in any practical sense. The discovery of (1) by Kaufmann (alias Mercator) in 1668 was a sensation. It was the point of departure for all the power-series expansions obtained, from then on, by Newton, Gregory, Leibnitz, and others.

Let me add that the equation in (2) is a quick consequence of Euler's celebrated result:

$$\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = \gamma$$

where γ is Euler's constant, whose value may be computed to many decimals, but whose rationality or irrationality is still one of the hardest unsolved problems in Mathematics.

from (1) is considerably more tricky than might be expected. One way is to use Taylor's theorem with Lagrange's remainder term. Another way is to let x in (1) tend to 1 (from below), and use a rather delicate argument known as Abel's. Although such an argument is sometimes given in the textbook, it has always daunted me to present it in class because I knew that the majority of students would not really appreciate it. Indeed, it is difficult enough to explain the very necessity of a proof of (2), once (1) is shown. In a sense, mathematics is forced on a defensive at such a juncture. Why should it be so hard to make such an obvious deduction, and since it is so hard, is it really worth the effort? In short, is this kind of mathematics useful?

Of course, mathematical reasoning learned in one situation may be useful in other similar or more difficult situations. Indeed, the example discussed above is just a junior-grade case of a whole slew of higher analysis. But this only goes to show that mathematics is useful for the study of more mathematics, perhaps of increasingly more abstruse and esoteric aspects. To interpret the question of utility in this way is clearly begging the question. Let me mention two more examples. You know that Fourier inauguarated his series in a treatise on the conduction of heat, surely one of the primary concerns of life on earth. Yet I read an article by a noted applied mathematician that so far as applications go, Fourier analysis of twice continuously differentiable function is sufficient. Thus current front-line research in this area, a rather difficult one, cannot be regarded as useful from his point of view. In probability theory, Markov chains were invented by Markov in a simple urn model. Similar models are now used extensively in economics, sociology, psychology as well as in the physical sciences. For such application a discrete-time, finite-state model is adequate, or occasionally the crudest kind of countable-state model known as birth-and-death processes. The requisite theory is an essentially solved problem by matrix methods. Thus nearly all the more general and subtler developments of the last thirty or forty years are largely irrelevant. I have been sometimes embarrassed by consultations on those applications which made me feel utterly useless. I should think that many of you had similar experiences in your respective areas of expertise. To use a current expression, mathematicians are "over-qualified" for applications.

Mathematicians are not alone in this peculiar situation. A few weeks ago I heard Richard Feynman say on television: "after the planets were worked out and the locations of a few stars, there is no more applications of astronomy... . With the big telescopes and all this effort, it has absolutely no application, none. ... In physics of high energy where there are again (as far as I can see) no applications, the highest energy physics trying to find out the fundamental laws of the tiniest dimension, a very expensive apparatus and I don't see any applications." He went on to say that some "foolish scientist" had in the past failed to foresee eventual applications, but added: "I'm going to make a prediction: there is no application whatsoever of what we are finding about high energy physics. So you can find me wrong in the future. But I don't think there is any application really for a long, long time, if ever." So, he did what he did in physics for "the pleasure of finding out" -- the title of the NOVA program which I watched 2/ . In so far as mathematics is used in these

2/ The transcript of Feynman's talk is available, from which the quotations are taken verbatim. The complete text is recommended to any reader who is interested in this discussion.

physical theories, to call it useful sounds like passing the buck. Another frequently held view is that although some mathematics. or physics for that matter, is obviously useless today, it may be useful some day. Feynman has anticipated such an argument in his quoted remarks above. An oft-cited example is the application of non-Euclidean geometry in relativity theory (never mind how useful the latter is). This contention has indubitable advantage that it can never be proved wrong, for no deadline is set on the eventual future. Hence it is an effective argument to extract financial support from society. But is it really likely that all, or even a good part of current mathematics will ever be found useful? Recently I asked Professor André Weil, who was lecturing at Stanford on Euler, this same question, and he gave a negative answer. In fact, he replied by first saying that the question had already been answered by G.H. Hardy in his book "A Mathematician's Apology" (Cambridge University Press, 1940; reprinted 1967). I know that book. When I first read it, I was very young and naive and could not have agreed more with Hardy. Incidentally, if you will permit me a digression here, it was Hardy's "A Course in Pure Mathematics", which was a reference in the freshman calculus course I took in Tsinghua University, Peking 1936, that started my conversion from physics to mathematics. To this day, I cannot get over the feeling that functions of n, rather than x, should be taught first in calculus as Hardy inculcated. (But try it in your class!) Let me quote some of Hardy's own words.

> Very little of mathematics is useful, and that little is comparatively dull.
> The 'real' mathematics of the 'real' mathematician, the mathematics of Fermat and Euler and Gauss and Abel and Riemann is almost wholly useless (and is as true of 'applied' as of 'pure' mathematics).
> Real mathematics must be justified as art if it can be justified at all.

Hardy gave two examples of 'real' mathematical theorems: Euclid's proof of the infinitude of primes and Pythagoras's proof of the irrationality of $\sqrt{2}$ Then he said, "neither has the slightest 'practical' importance". Here it should be clear that he meant "forever".

Well, I am aware that Hardy's "apology" would not do for most of us who lack his honesty and security. However we may think of his stance, the truth of his assertions is not in doubt. After I demonstrated the irrationality of $\sqrt{2}$ in an honors calculus course, I reminded the class that no matter how great our computers are or will be, nobody will ever see the square root of 2 on a print-out. But of course we do not need $\sqrt{2}$ in the real world.

If mathematics is largely useless, how can we justify its support by society? In earlier centuries men of independent means could indulge it at their own expense. Some were patronised just as great artists were. Euler was at the courts of Peter the Great, Frederick the Great and Catherine the Great. But we now must appeal to governments for funds to support "research projects". There is even talk in the learned societies about ways of "selling" research to the public, the incomprehending taxpayers or the ruling cliques. It would probably not do to speak one's mind so freely as Hardy did, but nor would some of us stoop to exaggerated claims, subterfuge and quackery. What should we do?

A realistic, tenable position may be the following. A certain amount of mathematics fairly crude and easy but nowadays rather widely spread, is useful for technology and other social functions. This kind of mathematics (which Hardy called "dull" and "trivial") must be taught at schools and colleges to a relatively large number of young men and women who will become engineers, physicists, chemists, biologists, actuaries, accountants, medical technicians, and the like. It is said that lawyers need some mathematical discipline to sharpen their wits in legal arguments. Poets and philosophers are known to study higher mathematics as a hobby. Politicians and economists "play with numbers". Judging from the frequency of slips of tongue in which "billion" was misspoken as "million" and vice versa, these citizens may need a refresher course on the significance of large numbers. (Both Hardy and Littlewood 3/ wrote about large and very small numbers.) In this era of trillion debt and megaton explosives I do not think this is a joke. Well, these people will go on to perform useful tasks for society, and they should be well trained. But in order to educate our youth well, we need teachers who can master their specialities. To teach engineering calculus one should know some advanced calculus, to teach advanced calculus, one should know some theory of real and complex functions. To teach linear programming one should know some abstract algebra and perhaps also some functional analysis. To teach even cookbook statistics one should know the basic principles of probability, and so on. A truly competent teacher should have a grasp on his subject matter better than he can probably transmit in class. Independent thinking is indispensable to the learning of mathematics, and this leads to the state of mind called "research" without which the knowledge remains passive. Last but not least, a good teacher should have a genuine love and enthusiasm for his subject. Is it then not natural that he will go beyond the call of duty, to dig deeper and roam farther? To return to the example above, having learned (1), a bright student would like to know whether one can put $x = 1$ there to get the beautiful result (2). And when he sees the problem there, he wants to solve it. The challenge presents itself and some will strive to meet it. Curiosity is the spur and aesthetic satisfaction the reward. Use is not the motive. Society in its care for the education of the young will support teachers, some of whom will become mathematicians because they have the capacity to do mathematics.

Professor A. Weil, with whom I discussed the matter last year, spoke of this view. No doubt he was thinking of the *ecole normale superieure*, which meant literally " advanced teachers' college" and the *ecole polytechique*, from which the majority of French mathematicians came.

The mathematician's role as instructor is extended or supplemented by his role as associate, consultant or, as I prefer to call it, a kind of general preceptor. Rudolf Kalman told me that his really useful filter was inspired by Wiener's not-so-useful prediction theory. If some mathematical idea is in the air, it may be picked up through a vague process of association and turned into visible use. By a change of metaphor, this process of association is sometimes described as "learning by osmosis". In my seminar on stochastic processes, given last quarter, there was an intelligent young man from the Department of Engineering-Economic Systems who was applying the rather abstruse theory of square integrable martingales to

3/ J.E.Littlewood, *A Mathemtician's Miscellany*, Mathuen 1953.

equilibrium problems of trading markets. (It may be wise for us not to inquire the usefulness of these issues.) The point is clear: a high level of achievement in one area tends to stimulate activities in related areas, as the study of Shakespeare may be useful to journalism.

When we stop to think about it, the historical blossoming of mathematics is indeed a wondrous, perhaps fortuitous event, more incredible than that of the arts whose roots lie closer to our daily life. The advanced civilization of ancient China did not develop mathematics to the level of the ancient Greeks or the seventeenth-eighteenth century Europeans. Is it because it was perceived as useless beyond a certain point, such as the computation of π and the numerical solutions of equations? 4/ It would seem that the mathematical mind is a more remote sort of human instinct than music, architecture and literature. Surely the number of citizens who can appreciate Euclid's and Pythagoras's proofs cited by Hardy is less than those who enjoy Bach, Michelangelo and Tu Fu. (I may be wrong there, but there are far more complicated mathematical theorems I have cited.)

While it is true that many mathematical fields sprung from humble utilitarian origins, it is incontestible that we now possess and are continuing to expand a body of knowledge and skill in mathematics infinitely broader and profounder than required for any practical purposes. How did this come about? Why, it happened naturally, as I tried to suggest above. A little mathematics is useful, some more must be taught and a few become mathematicians. It is a tribute to the human capacity that it did not stop at immediate or even forseeable goals. The intellect wants to go farther because it can. The mountain is climbed because it is there. After scaling one peak another appears on the horizon. It is indeed marvelous that the human gene pool contains so much mathematical genius and talent that we have advanced as far as we did. This is history. Short of disaster (made possible, partly by mathematics), we shall continue.

Any attempt to justify mathematics to "sell" it on utility alone, is selling it short. It is selling short the human capacity. As the cave dwellers made those paintings and the Mayan astronomers those buildings, so do we pursue a mathematical career to learn, to teach, to use it or help others use it whenever and wherever applicable, but above all to preserve and uphold this strange and fascinating human capacity called mathematics.

4/ During my trip to China in the summer of 1975 with my wife and son, I read in the official newspaper that the two great scientific achievements of that era were : the discovery of undersea petroleum deposits by geologist Lee, and the proof by Chen that every (large) even number is the sum of a prime and another number having at most two prime factors. This was the period when utilitarianism reigned supreme in China and all theoretic work was suspect. When we visited my old junior middle school in Hangchow, where I still remember the names of my good teachers, a young teacher of mathematics told me that he was confused as to what to teach and how to teach it. If Euclid's proof cited above does not have the slightest practical importance, it is inconceivable that Goldbach's conjecture (every even number is the sum of two primes), of which Chen's result is a weaker analogue, could ever have any use in China or anywhere else. Incidentally, it was also reported that China planned to do expensive research in high energy physics despite expert advice against it. Of course, prestige is quite useful in real life, but that is not the real "use" we are talking about here.