THEOREMS ABOUT TOUCHING CIRCLES

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1. Introduction

The Mathematical Association is an association, in the United Kingdom, of teachers and students of elementary mathematics. Its fundamental aim is to promote good methods of mathematical teaching, and many university mathematicians and other interested persons also belong to the Association, which publishes two journals, the long-established "Mathematical Gazette" and the more recent "Mathematics in School". At the Association's annual conference in Swansea in 1982, the newly elected President, Mr F. J. Budden, aired an interesting problem about touching circles. This was discussed, but not fully solved, by Tony Gardiner in the Mathematical Gazette [6]. We state the problem in the form of a theorem.

THEOREM 1. *Suppose that the circles a and b touch each other, and the line l is a common tangent; circle c_1 touches a, b and l as shown in figure 1; c_2 touches a, b and c_1; c_3 touches a, b and c_2, etc.; the radius of c_i is r_i and the distance of its centre from l is d_i. Then $d_2/r_2 = 7$. More generally, $d_i/r_i = 2i^2 - 1$.*

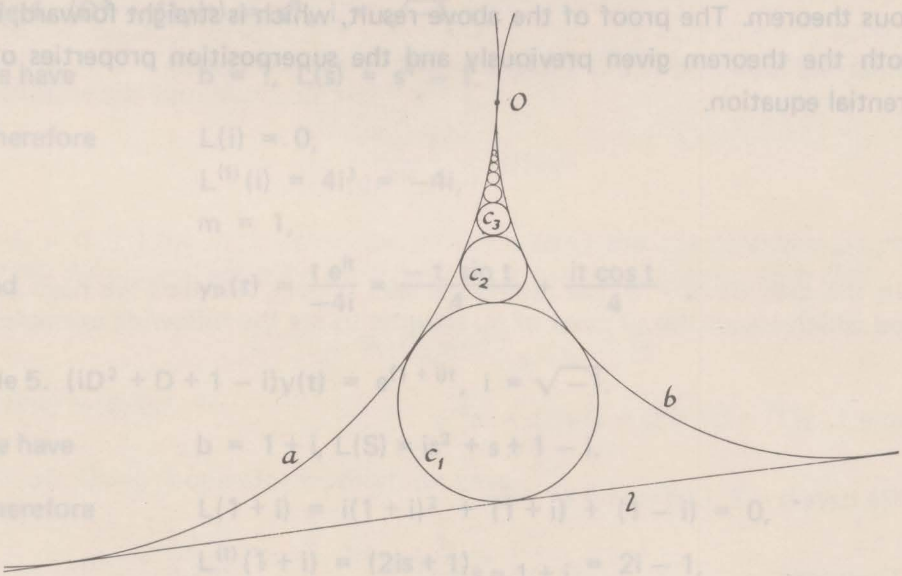

Fig. 1

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When Mr Budden set this problem in his "Problems Correspondence Course" (which goes out, via the Mathematical Association, to any interested people, who include school pupils, teachers, lecturers and retired people) he received various solutions, some using inversion, involving varying amounts of calculation. The proof to be given here is in one sense much simpler because it involves only a straightforward application of an important general theorem about inversion (Theorem 3); but of course calculations of some sort have to be used to prove the general theorem!

We shall discuss Theorem 3 in section 2 and shall use it in section 3 to prove Theorem 1, but the proof of Theorem 3 will be deferred until section 5. In section 3 we shall also calculate the radius of c_i in figure 1. In section 4 we shall prove corresponding results about a similar figure (figure 7).

Until the second half of section 5, a prior knowledge of only a few basic facts about inversion will be assumed.

2. The cosine between two circles

It is well know that, if two circles x and y intersect at an angle θ, and if we invert x and y to x' and y' (using any circle of inversion), then x' and y' intersect at the same angle θ. In other words:

THEOREM 2. *The angle between two intersecting circles is an inversive invariant.*

Suppose the circles x and y meet at P and Q, and have centres X and Y and radii r and s; write $d = XY$. Let θ denote the acute angle between x and y; then either $\angle XPY = \theta$ (figure 2a) or $\angle XPY = \pi - \theta$ (figure 2b) since the tangents at P are perpendicular to the radii. Hence by the cosine formula

$$\cos\theta = |(r^2 + s^2 - d^2)/2rs|.$$

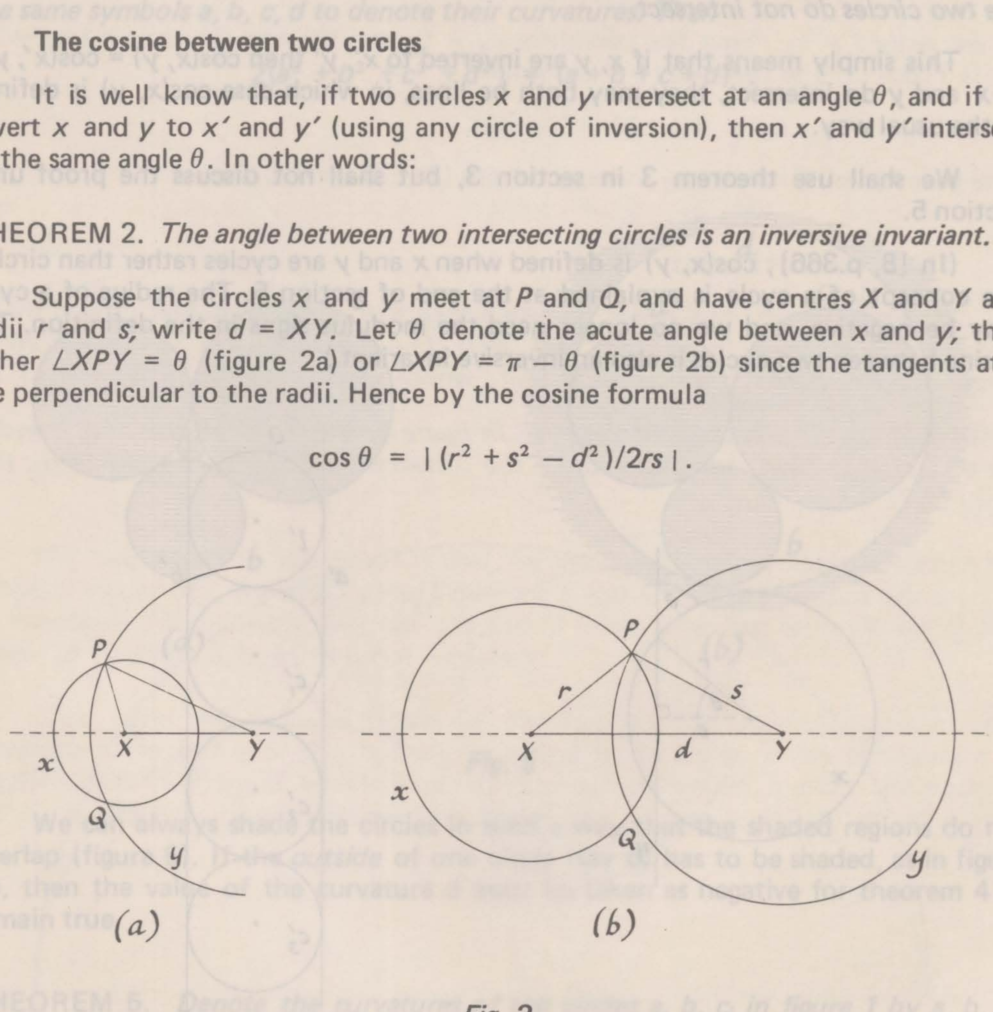

Fig. 2

The inverse of a circle is a line if the centre of inversion lies on the circle, and the inverse of a line is in general a circle; hence in inversive geometry lines are regarded as circles of a special type. Suppose the circle x and the line y intersect at an angle θ, as in figure 3; then $\cos\theta = e/r$.

Hence theorem 2 states that $|(r^2 + s^2 - d^2)/2rs|$ or e/r is an inversive invariant of two intersecting circles.

Now the expression $|(r^2 + s^2 - d^2)/2rs|$ or e/r is defined even if x and y do not intersect; we shall call it the *cosine* between x and y, and denote it by $\cos(x, y)$. (It can be regarded as the cosine of the "imaginary angle" between x and y when x and y do not intersect, in which case $\cos(x, y) > 1$, but there is no need to introduce this concept). It is not surprising that theorem 2 can be extended to theorem 3.

THEOREM 3. *The cosine between two circles is an inversive invariant, even when the two circles do not intersect.*

This simply means that if x, y are inverted to x', y' then $\cos(x, y) = \cos(x', y')$. If x and y do intersect, they may both be lines, in which case $\cos(x, y)$ is defined in the usual way.

We shall use theorem 3 in section 3, but shall not discuss the proof until section 5.

(In [8, p.366], $\cos(x, y)$ is defined when x and y are cycles rather than circles; the concept of a cycle is explained at the end of section 5. The radius of a cycle may be negative, and we no longer need the modulus signs in the definition. The cosine between two cycles is also an inversive invariant.)

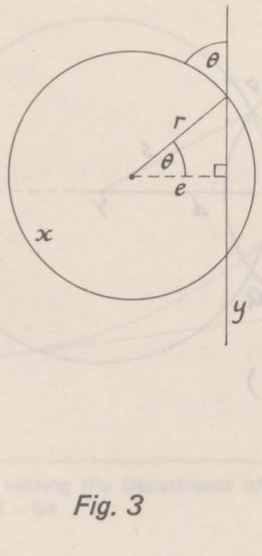

Fig. 3

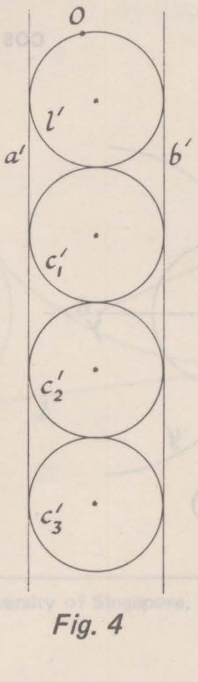

Fig. 4

3. Investigation of figure 1

PROOF OF THEOREM 1. Invert figure 1 with respect to O; we obtain figure 4, since circles touching at O invert to parallel lines, and touching circles or lines invert to touching circles or lines. Then

$$d_i/r_i = \cos(c_i, l) = \cos(c'_i, l') = |1^2 + 1^2 - (2i)^2|/2 = 2i^2 - 1.$$

To calculate the radius of the circle c_i we can use a theorem that was first proved by Descartes. Much has been written about this theorem, and many proofs given; see [4, 5] for further references. The *curvature* of a circle of radius r is defined to be $1/r$; a line has curvature 0.

THEOREM 4 (Descartes' Circle Theorem). *Let a, b, c, d be four circles each touching the other three (and not all touching each other at the same point), and use the same symbols a, b, c, d to denote their curvatures. Then*

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

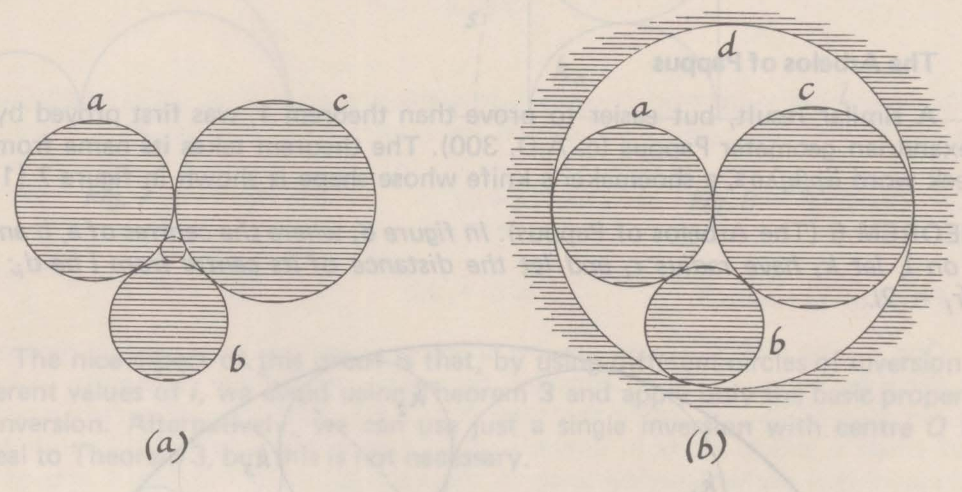

Fig. 5

We can always shade the circles in such a way that the shaded regions do not overlap (figure 5). If the *outside* of one circle (say d) has to be shaded, as in figure 5b, then the value of the curvature d must be taken as negative for theorem 4 to remain true.

THEOREM 5. *Denote the curvatures of the circles a, b, c_i in figure 1 by a, b, c_i. Then* $c_i = i^2(a + b) + 2i\sqrt{(ab)}$.

PROOF. The curvature of l is 0. Hence by Descartes' theorem

$$2(a^2+b^2+c_1^2) = (a+b+c_1)^2 \quad (1)$$

$$\therefore\ c_1^2 - 2(a+b)c_1 + (a-b)^2 = 0$$

$$\therefore\ c_1 = (a+b) + 2\sqrt{(ab)}. \quad (2)$$

(There is a second circle touching a, b, l; its curvature is $(a+b) - 2\sqrt{(ab)}$.)

Also by Descartes' theorem

$$2(a^2+b^2+c_i^2+c_{i+1}^2) = (a+b+c_i+c_{i+1})^2 \quad (3)$$

and $$2(a^2+b^2+c_{i+1}^2+c_{i+2}^2) = (a+b+c_{i+1}+c_{i+2})^2. \quad (4)$$

Subtracting (3) from (4), we have

$$2(c_{i+2}^2 - c_i^2) = c_{i+2}^2 - c_i^2 + 2(a+b+c_{i+1})(c_{i+2}-c_i)$$

$$\therefore\ c_{i+2} + c_i = 2(a+b+c_{i+1}). \quad (5)$$

If we write $l = c_0$, then the curvature $c_0 = 0$, and (3) and (5) remain true when $i = 0$. From (2) and (5) with $i = 0$ we obtain

$$c_2 = 4(a+b) + 4\sqrt{(ab)}. \quad (6)$$

The general result follows by induction from (2), (6) and (5).

4. The Arbelos of Pappus

A similar result, but easier to prove than theorem 1, was first proved by the Alexandrian geometer Pappus (c. A.D. 300). The theorem takes its name from the Greek word ἄρβηλος, a shoemaker's knife whose shape is shown in figure 7 [1, 9].

THEOREM 6 (The Arbelos of Pappus). *In figure 6, where the centres of a, b and k_0 lie on l, let k_i have radius r_i and let the distance of its centre from l be d_i; then $d_i/r_i = 2i$.*

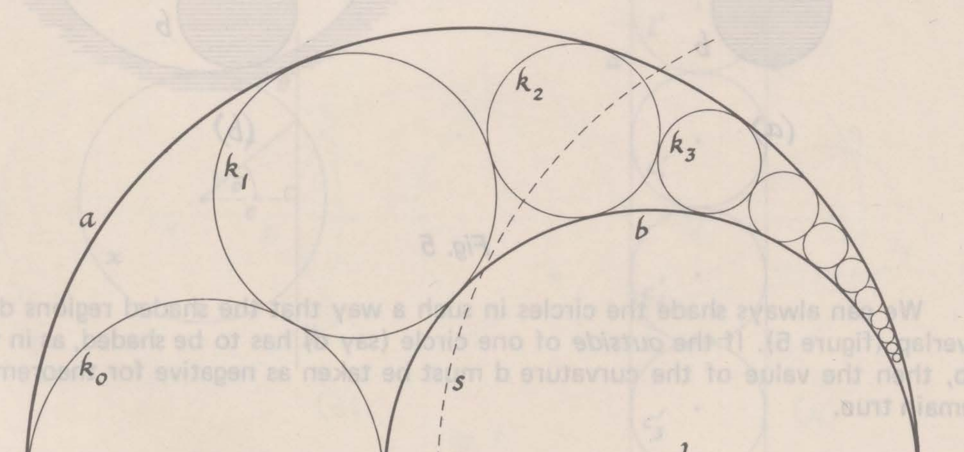

Fig. 6

PROOF. There exists a circle s with centre O, orthogonal to k_i (in figure 6, $i = 2$). Invert figure 6 with respect to s. We obtain figure 8; but k_i and l are inverted to themselves, so clearly from figure 8 $d_i/r_i = 2i$.

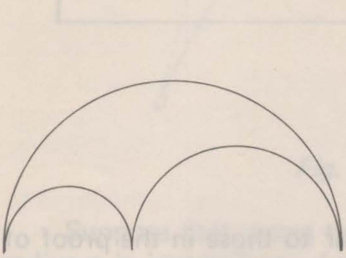

Fig. 7

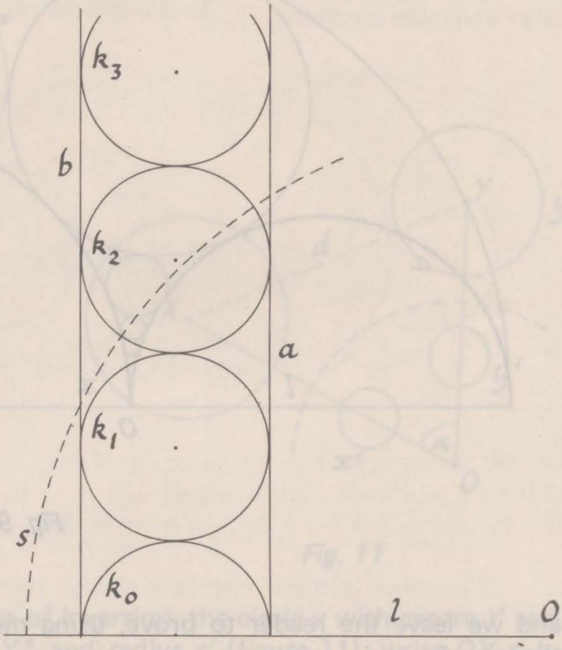

Fig. 8

The nice aspect of this proof is that, by using different circles of inversion for different values of i, we avoid using Theorem 3 and apply only the basic properties of inversion. Alternatively, we can use just a single inversion with centre O and appeal to Theorem 3, but this is not necessary.

Once again we can use Descartes' theorem to find an expression for the curvature k_i in terms of the curvatures a and b. In figure 6 the curvature a is negative, but in figure 9, which illustrates the same theorem, a and b are positive but k_0 is negative. In all cases we find, taking the correct signs into account, that

$$k_0 = -ab / (a + b),$$

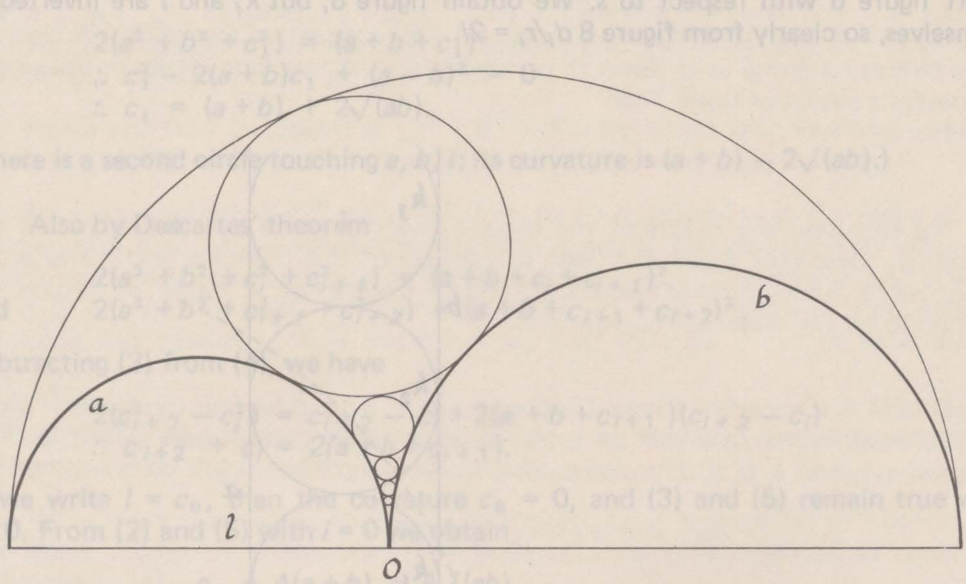

Fig. 9

and we leave the reader to prove, using methods similar to those in the proof of theorem 5:

THEOREM 7. $k_i = i^2(a+b) - ab/(a+b)$.

5. Proof of Theorem 3

One of the difficulties of proving theorem 3 by an "elementary" method is that there are various cases to consider, depending on whether the centre of inversion lies outside, on, or inside the two circles. We shall content ourselves with a proof when x and y are both circles and the centre of inversion lies outside both of them; this is perhaps unfair since in the application to theorem 1 one of the circles is a line, but this can be regarded as a limiting case.

In figure 10, the circle x with centre X and radius r is inverted to x' with centre X^* and radius r', using a circle of inversion with centre O and radius k; write $OX = a$, $OX^* = a'$. Then $OP.OP' = k^2$, i.e. $\sqrt{(a^2 - r^2)}\sqrt{(a'^2 - r'^2)} = k^2$; also $OP/PX = OP'/P'X^*$, i.e. $\sqrt{(a^2 - r^2)}/r = \sqrt{(a'^2 - r'^2)}/r'$. We deduce that

$$a^2 - r^2 = k^2 r/r'. \tag{7}$$

Also $OX/XP = OX^*/X^*P'$, i.e.

$$a/r = a'/r'. \tag{8}$$

(These formulae remain true when O lies inside x, if we then regard r' and a' as being negative.)

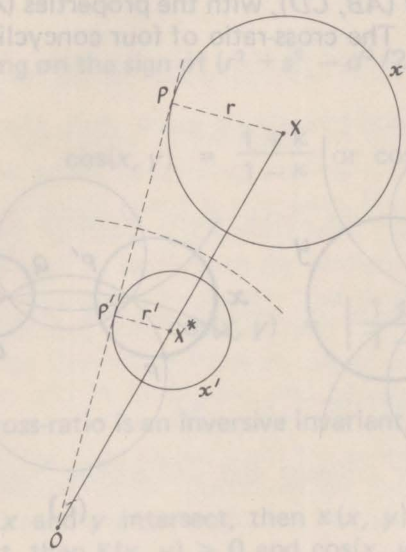

Fig. 10

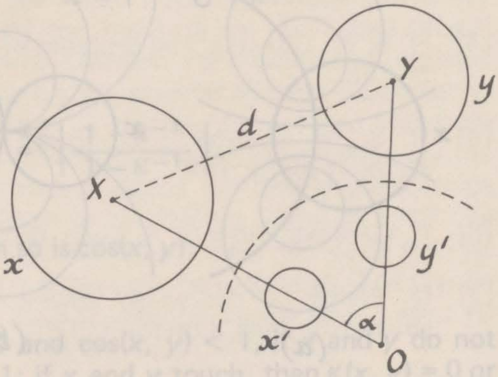

Fig. 11

Suppose that, using the same circle of inversion, the circle y with centre Y and radius s is inverted to y' with centre Y^* and radius s' (figure 11); write $OY = b$, $OY^* = b'$. Then

$$b^2 - s^2 = k^2\, s/s' \tag{9}$$

and

$$b/s = b'/s'. \tag{10}$$

Now

$$\cos(x, y) = |(r^2 + s^2 - d^2)/2rs|$$

$$= \left| \frac{r^2 + s^2 - a^2 - b^2 + 2ab\cos\alpha}{2rs} \right| = \left| \frac{-k^2\, r/r' - k^2\, s/s'}{2rs} + \frac{ab\cos\alpha}{rs} \right|$$

$$= \left| \frac{-k^2\,(rs' + sr')}{2rr'ss'} + \frac{a}{r} \cdot \frac{b}{s}\cos\alpha \right| .$$

This expression remains unchanged when we replace r, s, r', s' by r', s', r, s respectively; hence $\cos(x, y) = \cos(x', y')$.

Theorem 3 emerges in a much more satisfactory manner, avoiding the long algebraic proof and the consideration of special cases, if we use the important and basic concept of the *cross-ratio* of four concyclic points to define the cosine between two circles. Various texts, such as [7], give the definition and basic properties of cross-ratios and other concepts associated with circles and inversion.

Associated with an ordered set A, B, C, D of four concyclic points is a real number, called their cross-ratio, denoted by (AB, CD), with the properties $(AB, CD) = (BA, DC) = (AB, DC)^{-1} = (BA, CD)^{-1}$. The cross-ratio of four concyclic points is an inversive invariant.

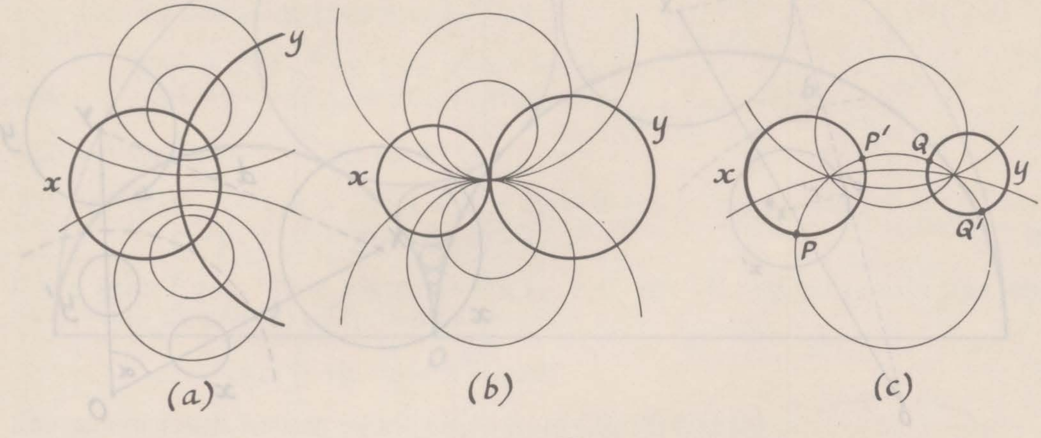

Fig. 12

Given two distinct circles x and y, there is an infinity of circles orthogonal to both, forming the *coaxial system orthogonal to x and y* (figure 12). If any circle c of this coaxial system meets x at P, P' and y at Q, Q' (figure 12c), then the cross-ratio (PP', QQ') is independent of c. We call this the *cross-ratio of x and y*. Since P and P' could be labelled in the opposite order, and similarly for Q and Q', we have to allow two cross-ratios, (PP', QQ') and $(PP', Q'Q)$; we denote them by $\kappa(x, y)$ and $\kappa(x, y)^{-1}$.

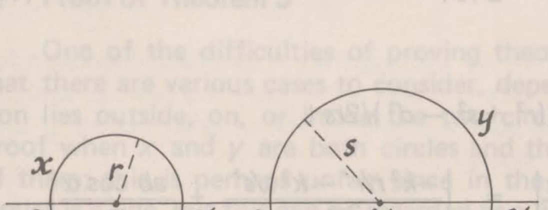

Fig. 13

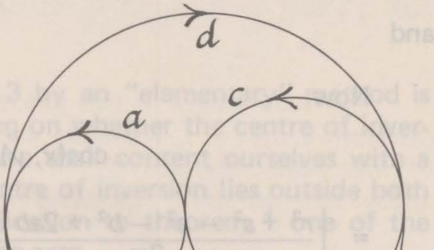

Fig. 14

Taking c to be a line (figure 13) and using the formula

$$(PP', QQ') = PQ.P'Q'/PQ'.P'Q,$$

we find (using the previous notation) that

$$\kappa = (PP', QQ') = (d^2 - r^2 - s^2 + 2rs)/(d^2 - r^2 - s^2 - 2rs)$$

$$= \frac{\cos(x,y) - 1}{\cos(x,y) + 1} \text{ or } \frac{\cos(x,y) + 1}{\cos(x,y) - 1}$$

depending on the sign of $(r^2 + s^2 - d^2/2rs$. Hence either

$$\cos(x,y) = \frac{1+\kappa}{1-\kappa} \text{ or } \cos(x,y) = \frac{\kappa+1}{\kappa-1} = \frac{1+\kappa^{-1}}{1-\kappa^{-1}}.$$

Hence

$$\cos(x,y) = \left|\frac{1+\kappa}{1-\kappa}\right| = \left|\frac{1+\kappa^{-1}}{1-\kappa^{-1}}\right|.$$

Since cross-ratio is an inversive invariant, then so is $\cos(x, y)$.

If x and y intersect, then $\kappa(x, y) < 0$ and $\cos(x, y) < 1$, if x and y do not intersect, then $\kappa(x, y) > 0$ and $\cos(x, y) > 1$; if x and y touch, then $\kappa(x, y) = 0$ or ∞ and $\cos(x, y) = 1$. If x and y are orthogonal, then $\kappa(x, y) = -1$ and $\cos(x, y) = 0$.

If x and y do not intersect, we define the *inversive distance* between them, $\mathrm{d}(x, y)$, to be $\cosh^{-1}\cos(x, y)$ [2, 3]. This distance function satisfies the property that if x, y, z are coaxial, with y between x and z, then

$$\mathrm{d}(x,y) + \mathrm{d}(y,z) = \mathrm{d}(x,z),$$

and is useful in the Poincare model of the hyperbolic plane.

The sign of the radius and the curvature of a circle is automatically taken care of if we use cycles rather than circles. A *cycle* is a circle with a direction on it indicated by an arrow. An anticlockwise circle has positive radius and a clockwise circle has negative radius. All the touching circles in this article must be replaced by *anti-touching* cycles, with the arrows pointing in opposite directions at the point of tangency. Alternatively, a cycle can be regarded as either the interior or the exterior of a circle: compare figure 14 with figure 5b.

The formula $\cos(x, y) = (r^2 + s^2 - d^2)/2rs$, without modulus signs, is used for the cosine between two cycles [8, p. 366]; this is an inversive invariant. Unfortunately there seems to be no way of using cross-ratios to define the cosine between two cycles: the previous definition cannot distinguish between positive and negative cosines.

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