SOLUTIONS TO PART A

1. Clearly,

$$878787878787 = 87 \times 10101010101$$
and $$787878787878 = 78 \times 10101010101.$$

Since gcd(87, 78) = 3, the gcd of the two given numbers is 30303030303. Thus the correct answer is (e).

2. Let $A = 10^m - 1$ and $B = 10^n - 1$. Then $AB = 10^{m+n} - (10^m + 10^n) + 1$. Since $10^m + 10^n$ is of the form 10 . . . 010 . . . 0, $AB = (10^{m+n} + 1) - (10^m + 10^n)$ is of the form 9 . . . 989 . . . 90 . . . 01. So the answer is (b). Note that the result is also true if m = n.

3. (d) is false, for let a = 8, b = 4. Then $64 = a^2 \mid b^3 = 64$. However, $a \nmid b$.

4. The required number of zeros is exactly equal to the highest power of 5 in 100!. Hence the answer is:

$$[\frac{1000}{5}] + [\frac{1000}{25}] + [\frac{1000}{125}] + [\frac{1000}{625}]$$
$$= 200 + 40 + 8 + 1$$
$$= 249.$$

The correct answer is (d).

5. A positive integer n is relatively prime to 12 if and only if it is not divisible by 2 or 3. The probability that n is divisible 2, 3 or 6 is $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ respectively. Hence the required probability is

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1}{3}.$$

The correct answer is (d).

6. We have equilateral triangles of 5 increasing sizes (say, sizes 1, 2, 3, 4, 5 respectively). The numbers of triangles of sizes 1, 2, 3, 4, 5 are respectively:

25, 13, 6, 3 and 1.

Hence the required number is

$$25 + 13 + 6 + 3 + 1 = 48.$$

The correct answer is (b).

7. There are altogether 6 matches. The total number of points is thus 12. Let the points awarded to Singapore, Malaysia, Thailand and Indonesia be S, M, T, I respectively. Then S + M + T + I = 12, with S, M, T, I all distinct, S, I even and

M, T odd. So the only possibilities are: $12 = 0 + 3 + 4 + 5$ and $12 = 0 + 1 + 5 + 6$. The second case is impossible; for if every team beat Indonesia, then S, M, T $\geqslant 2$. Hence we must have the first case, from which we conclude that S = 4, (2nd place). Thus the correct answer is (c).

8. We have

$$= \int_0^1 (x - \tfrac{1}{2})^3 f(x)dx$$

$$= \int_0^1 x^3 f(x)dx - \frac{3}{2}\int_0^1 x^2 f(x)dx + \frac{3}{4}\int_0^1 xf(x)dx - \frac{1}{8}\int_0^1 f(x)dx$$

$$= 1.$$

If $M < 32$, then

$$1 = \int_0^1 (x - \tfrac{1}{2})^3 f(x)dx < 32 \int_0^1 |x - \tfrac{1}{2}|^3 dx = 1, \text{ a contradiction.}$$

Hence we conclude that $M \geqslant 32$. The correct answer is (e).

9. The hour-hand moves through θ°, taking $\dfrac{12\theta}{360}$ hrs. The minute-hand moves through $360^\circ + \theta^\circ$, taking $\dfrac{360 + \theta}{60 \times 60}$ hrs.

We have: $\dfrac{12\theta}{360} = \dfrac{360 + \theta}{60 \times 60}$.

Hence $12\theta = 360 + \theta$

Therefore $\theta = \dfrac{360}{11} = 32\dfrac{8}{11} > 32\dfrac{1}{2}$.

The correct answer is (a).

10. Let the number of socks of each colour be n. Then the number of ways of choosing three socks is $\binom{3n}{3}$. The number of ways of choosing three socks of different colours is $\binom{n}{1}^3$. Hence the required probability is

$$p = 1 - \binom{n}{1}^3 \Big/ \binom{3n}{3}$$

$$= 1 - n^3 \frac{1 \cdot 2 \cdot 3}{3n(3n-1)(3n-2)}$$

$$= 1 - \frac{2}{(3 - \frac{1}{n})(3 - \frac{2}{n})}$$

$$= \frac{7}{9}, \text{ as } n \to \infty.$$

The correct answer is (d).

SOLUTIONS TO PART B

1. M_2 is expressible in the form $m_2 = mq_2 + 1$ where q_2 is an integer. In fact $q_2 = m - 1$. Also if for any k, $m_k = mq_k + 1$ where q_k is an integer, then

$$\begin{aligned} m_{k+1} &= m_k^2 - m_k + 1 \\ &= m(mq_k^2 + q_k) + 1 \\ &= mq_{k+1} + 1 \end{aligned}$$

where $q_{k+1} = mq_k^2 + q_k$ is an integer.

It follows by mathematical induction that for every $i > 1$, $m_i = mq_i + 1$ whence m_i is not divisible by m.

2. Let f be in p such that $f(g(x)) = g(f(x))$ for each g in p. Let $g(x) = x + h$, where $h \neq 0$. Observe that $f(g(x)) = f(x + h)$ and $g(f(x)) = f(x) + h$. Thus $f(x + h) = f(x) + h$, i.e.,

$$\frac{f(x+h) - f(x)}{h} = 1.$$

Hence $\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1$

and so $f(x) = x + c$ where c is a constant.

Now let g = 0. Then $g(f(x)) = 0$ and $f(g(x)) = 0 + c$, which imply that c = 0. We thus conclude that $f(x) = x$.

It is easily seen that $f(x) = x$ satisfies $f(g(x)) = g(f(x))$ for all g in p.

Hence $\{ f \in p \mid f(g(x)) = g(f(x)) \text{ for all } g \in p \} = \{ I \}$

where $I(x) = x$.

3. Let $S_n = (1 + \frac{1}{1!}) + (\frac{1}{2} + \frac{1}{2!}) + \ldots + (\frac{1}{n} + \frac{1}{n!})$. Then $(n-1)!\, S_n = k + \frac{(n-1)! + 1}{n}$ for some integer k. If n divides $(n-1)! + 1$, take any prime division p of n. Then $p \leqslant n - 1$, and so p divides $(n-1)!$. But then $p \mid 1$, which is impossible. Hence $((n-1)! + 1)/n$ is not an integer, and thus S_n is not an integer.

4. For $k \geqslant m + 2$, we have

$$\begin{aligned} 0 < \frac{n}{2^k} + \frac{1}{2} &= a_m 2^{m-k} + a_{m-1} 2^{m-k-1} + \ldots + a_0 2^{-k} + \frac{1}{2} \\ &\leqslant 2^{m-k} + 2^{m-k-1} + \ldots + 2^{-k} + \frac{1}{2} \\ &\leqslant 2^{-2} + 2^{-3} + \ldots + 2^{-k} + \frac{1}{2} \\ &< 1. \end{aligned}$$

Hence $[\frac{n}{2^k}+\frac{1}{2}] = 0.$

It is easy to see that

$$\frac{1}{2}+\frac{n}{2^k}=\frac{1}{2}+\frac{a_m}{2}+\ldots+\frac{a_0}{2^m}$$

and so $[\frac{1}{2}+\frac{n}{2^k}] = \begin{cases}0 & \text{if } a_m = 0\\ 1 & \text{if } a_m = 1,\end{cases}$ i.e. $[\frac{1}{2}+\frac{n}{2^k}] = a_m.$

We shall next show that

$$[\frac{1}{2}+\frac{n}{2^k}] = a_m 2^{m-k}+\ldots+a_k+a_{k-1} \text{ for } k \leqslant m.$$

Observe that

$$\frac{n}{2^k}+\frac{1}{2} = (a_m 2^{m-k}+\ldots+a_k)+(\frac{a_{k-1}}{2}+\frac{a_{k-2}}{2^2}+\ldots+\frac{a_0}{2^k}+\frac{1}{2})$$

Since $\frac{1}{2}\leqslant\frac{a_{k-1}}{2}+\frac{a_{k-2}}{2^2}+\ldots+\frac{a_0}{2^k}+\frac{1}{2})$

$$\leqslant\frac{a_{k-1}}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2}<\frac{a_{k-1}}{2}+1,$$

We have $[\frac{a_{k-1}}{2}+\frac{a_{k-2}}{2^2}+\ldots+\frac{a_0}{2^k}+\frac{1}{2}] = \begin{cases}0 & \text{if } a_{k-1} = 0\\ 1 & \text{if } a_{k-1} = 1\end{cases}$

Thus $[\frac{1}{2}+\frac{n}{2^k}] = a_m 2^{m-k}+\ldots+a_k+a_{k-1}$ if $k \leqslant m.$

Finally $\sum_{k=1}^{n} [\frac{n}{2^k}+\frac{1}{2}]$

$$\begin{aligned}
&= a_m + (a_m + a_{m-1}) + (a_m 2 + a_{m-1} + a_{m-2}) + (a_m 2^2 + a_{m-1} 2 + a_{m-2} + a_{m-3}) + \ldots + (a_m 2^{m-1} + \ldots + a_1)\\
&= a_m (1+1+2+2^2+\ldots+2^{m-1}) + a_{m-1} (1+1+2+2^2+\ldots+2^{m-2}) + \ldots + a_0\\
&= a_m 2^m + a_{m-1} 2^{m-1} + \ldots + a_0\\
&= n.
\end{aligned}$$

5. Observe that

$$
\begin{aligned}
X_n &= 2^{E_{n-1}} - 2^{E_{n-2}} \\
&= 2^{E_{n-2}}\left\{2^{(E_{n-1} - E_{n-2})} - 1\right\} \\
&= 2^{E_{n-2}}\,(2^{X_{n-1}} - 1)
\end{aligned}
$$

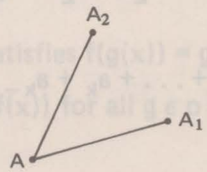
Thus $$\frac{X_n}{X_{n-1}} = 2^{X_{n-2}}\left(\frac{2^{X_{n-1}} - 1}{2^{X_{n-2}} - 1}\right)$$

But if a/b then $(2^a - 1)/(2^b - 1)$.

This gives an easy proof of the result by induction.
(Note that the result is true for n = 3)

6. Suppose on the contrary that a closed polygon $A_1, A_2, \ldots, A_k$ $(A_1 = A_{k+1})$ is formed. Then for each i = 1, 2, . . ., k, either A_i is the closest neighbour of A_{i+1} or A_{i+1} is the closest neighbour of A_i. Without loss of generality, one can assume that A_1 is the closest neighbour of A_2. Then $A_1A_2 < A_2A_3$. So A_2 is the closest neighbour of A_3 and $A_2A_3 < A_3A_4$. Continuning this process, we get: A_i is the closest neighbour of A_{i+1} for each i = 1,2, . . ., k, and $A_1A_2 < A_2\ A_3 < \ldots < A_kA_1 < A_1A_2$, which is impossible. Therefore there is no closed polygon.

Consider



Suppose A is the closest neighbour of both A_1 and A_2. Then $A_1A_2 > A_1A$ and $A_1A_2 > A_2A$.

Suppose A is the closest neighbour of only one of A_1 and A_2, say A_1. Then A_2 is the closest neighbour of A and we have

$$AA_2 < AA_1 < A_1A_2.$$

Hence $\angle A^1AA^2 > 60^\circ$. Therefore A can be joined to at must 5 other points.