

Nagayoshi Iwahori
University of Tokyo

When one looks at mathematical phenomena or theories from various points of view instead of from only one angle, one usually gets some unexpected harvests and discoveries. There are so many such examples in the history of mathematics.

- (1) Jacobi and others consciously applied the theory of complex analytic functions, especially the theory of elliptic functions, to number theory.
- (2) The proof of Dirichlet's theorem on the existence of an infinite number of primes in the arithmetic progression $m, m + k, m + 2k, \dots$, where m, k are coprime positive integers, uses very heavily the properties of the L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a character.

- (3) The Dedekind η -function has recently been taken up by Macdonald, and the expansion formula for the powers of η is reduced to a classical Weyl denominator formula through the use of infinite-dimensional Lie algebras, in particular the Kac-Moody Lie algebras.

Let us now take up some concrete problems and look at their discrete and continuous versions.

Proposition 1 (via Louis Chen)

If a smooth function f on $[0, 1]$ satisfies $f(0) = f(1) = 0$, $\int_0^1 f(x) dx = 1$, then there exists $\xi \in (0, 1)$ such that $|f'(\xi)| > 4$. The value 4 is best possible.

The discrete version of this proposition is as follows.

Proposition 2

$$\inf_{\substack{x_1 + \dots + x_n = 1 \\ x_1 = x_n = 0}} \left(\sup_{1 \leq i \leq n-1} |x_i - x_{i+1}| \right) = \begin{cases} \frac{1}{k(k-1)}, & n = 2k, \\ \frac{1}{k^2}, & n = 2k+1. \end{cases}$$

*Text of a lecture organized by the Department of Mathematics, National University of Singapore, and the Singapore Mathematical Society on 10 February 1983. Notes taken by Y. K. Leong.

We give the following two concrete examples.

Example 1

The position $(3, 17, x)$ is a winning position for the second player if and only if $x = 18$.

To see this, write

$$3 = 1 + 1.2,$$

$$17 = 1 + 1.2^4,$$

$$x = \epsilon_0 + \epsilon_1 \cdot 2 + \epsilon_2 \cdot 2^2 + \epsilon_3 \cdot 2^3 + \epsilon_4 \cdot 2^4 + \dots,$$

where $\epsilon_i \in \{0, 1\}$, $i = 0, 1, \dots$. The criterion implies that $\epsilon_1 = \epsilon_4 = 1$ and all the other ϵ_i are 0, i.e. $x = 2 + 16 = 18$.

Example 2

For the starting position $(37, 100, 200)$, what is the best move for the first player?

First express in 2-adic expansion:

$$37 = \textcircled{1} + 4 + 32,$$

$$100 = 4 + 32 + 64,$$

$$200 = \textcircled{8} + 64 + \textcircled{128}$$

In each column of the expansions, identify the unpaired term (which we call a "chonga", meaning "he who has no opponent" in Japanese). The winning strategy for the first player is to alter the number containing the largest chonga (or "kill the largest chonga") in such a way that the resulting numbers contain no chonga. The new number that arises after alteration represents the number of stones which the first player should remove from that pile. In this example, 200 contains the largest chonga (128). We alter this number (200) so that the resulting number is $1 + 64$. Thus the first player should remove 135 stones from the third pile:

$$(37, 100, 200) \rightarrow (37, 100, 65).$$

(II) Wythoff's game. Played by 2 players. Given 2 piles of stones, denoted by $(a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. The first player can take any number of stones out of any one pile or he can take the same number of stones out of the 2 piles. The same rule applies to the second player. Moves alternate between the players and there is no passing. The winner is the one who clears the board.

The starting position (a, b) which is a winning position for the second player is given in Table 1, where "o" denotes a winning position and only the positions (a, b) , $a \leq b$, are shown (there is symmetry about the diagonal line).

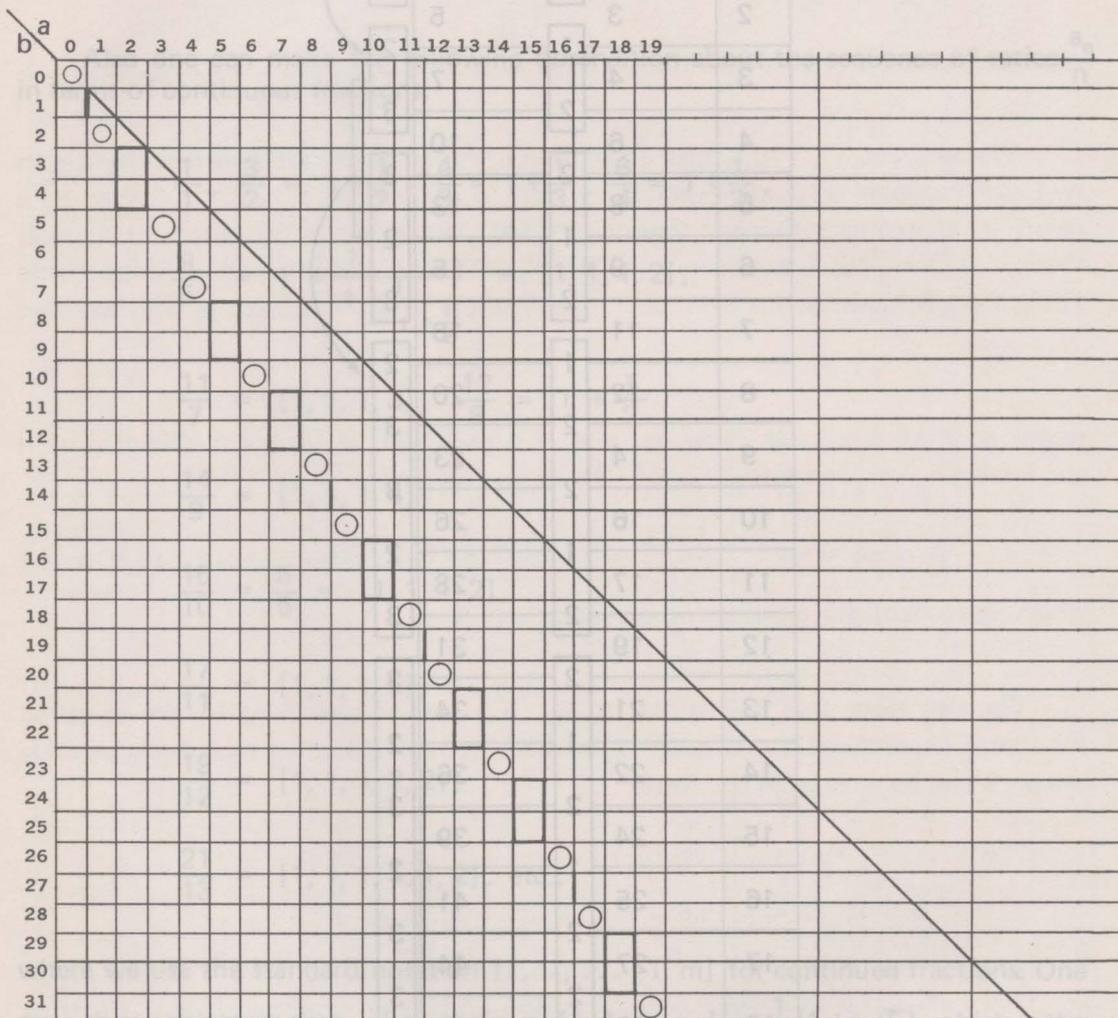


Table 1

From this one can observe the following phenomenon among the sequence of winning positions (a_n, b_n) , $a_n \leq b_n$. In Table 2, the boxed numbers denote the respective differences between consecutive a_n and b_n . If one groups these

differences into blocks as shown in Table 2, one realises that each block is obtained as a composite of the preceding two blocks in their respective order. One may say that the block formation obeys the "Fibonacci law". Recall that the sequence of Fibonacci numbers, F_1, F_2, F_3, \dots , is defined by

$$F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 3.$$

Also one can make the following observation about the sequence of ratios $\frac{a_n}{n}$ in terms of continuous fractions:

$$\frac{1}{1}, \frac{3}{2} = 1 + \frac{1}{2}, \frac{4}{3} = 1 + \frac{1}{3}, \frac{6}{4} = 1 + \frac{1}{2},$$

$$\frac{8}{5} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}},$$

$$\frac{11}{7} = [1, 1, 1, 3], \quad \frac{12}{8} = 1 + \frac{1}{2}$$

$$\frac{14}{9} = [1, 1, 1, 4],$$

$$\frac{16}{10} = \frac{8}{5} = [1, 1, 1, 2],$$

$$\frac{17}{11} = [1, 1, 1, 5],$$

$$\frac{19}{12} = [1, 1, 1, 2, 2],$$

$$\frac{21}{13} = [1, 1, 1, 1, 1, 2], \text{ etc.,}$$

where we use the standard notation $[1, 1, \dots, 1, m]$ for continued fractions. One may then conjecture that $\lim_{n \rightarrow \infty} a_n/n = [1, 1, 1, \dots] = \frac{1}{2}(1 + \sqrt{5})$, which is the well-known "golden ratio". For large n , we would then have $a_n \doteq \frac{1}{2}(1 + \sqrt{5})n$; in other words, the points (n, a_n) lie almost on the line $y = \frac{1}{2}(1 + \sqrt{5})x$. So one may conjecture that

$$a_n = [\frac{1}{2}(1 + \sqrt{5})n],$$

where $[x]$ denotes the greatest integer not exceeding x . Indeed this is so.

For completeness, we will describe a constructive method for the a_n, b_n :

$$a_0 = 0, b_0 = 0; a_1 = 1, b_1 = 2;$$

$$\begin{cases} a_n = \min (\mathbb{N} - \{a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1}\}), \\ b_n = a_n + n, \end{cases}$$

where \mathbb{N} is the set of positive integers.

The sets $\{a_n\}, \{b_n\}$ give a partition of \mathbb{N} . So if one assumes that there exist two real numbers α, β such that $a_n = [\alpha n], b_n = [\beta n], n = 1, 2, \dots$, then by a result in elementary number theory (see I.M. Vinogradov, "Elements of number theory", Dover, 1954, Problem 3, p. 29; cf. Interschool Mathematical Competition 1982, Part B, Contest Problem 2 – this Medley, Vol. 10 (1982), p. 58), we have

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

where α, β are irrational. Now, $b_n = a_n + n$ gives $[\beta n] = [\alpha n] + n$. Dividing by n and letting $n \rightarrow \infty$, one gets $\beta = \alpha + 1$. Hence one obtains

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \beta = \frac{1}{2}(3 + \sqrt{5}),$$

which agrees with the previous observation.

We also note that the triple (n, a_n, b_n) can be described by the so-called Fibonacci decomposition of positive integers. It is well-known that every positive integer k can be written in the form

$$\left. \begin{aligned} k &= F_{j_1} + \dots + F_{j_r}, \\ j_1 - j_2 &\geq 2, \dots, j_{r-1} - j_r &\geq 2, j_r &\geq 2. \end{aligned} \right\} \dots \dots \dots (1)$$

Furthermore the indices j_1, \dots, j_r are uniquely determined by k . The expression in (1) is called the Fibonacci decomposition of k . For example, if $k = 10$, then $k = 8 + 2 = F_6 + F_3$.

Now, if $n = F_{2m}$, then we have

$$\begin{cases} a_n = F_{2m} + F_{2m-2} + F_{2m-4} + \dots + F_2, \\ b_n = F_{2m+1} + F_{2m-1} + F_{2m-3} + \dots + F_3. \end{cases}$$

Otherwise, if $n = F_{j_1} + F_{j_2} + \dots + F_{j_r}, r \geq 2$, satisfying (1), then we have

$$\begin{cases} a_n = F_{j_1+1} + F_{j_2+1} + \dots + F_{j_r+1}, \\ b_n = F_{j_1+2} + F_{j_2+2} + \dots + F_{j_r+2}. \end{cases}$$

Table 3 illustrates these facts.

n	a_n	b_n
$1 = F_2$	$1 = F_2$	$2 = F_3$
$2 = F_3$	$3 = F_4$	$5 = F_5$
$3 = F_4$	$4 = F_4 + F_2$	$7 = F_5 + F_3$
$4 = F_4 + F_2$	$6 = F_5 + F_2$	$10 = F_6 + F_3$
$5 = F_5$	$8 = F_6$	$13 = F_7$
$6 = F_5 + F_2$	$9 = F_6 + F_2$	$15 = F_7 + F_3$
$7 = F_5 + F_3$	$11 = F_6 + F_4$	$18 = F_7 + F_5$
$8 = F_6$	$12 = F_6 + F_4 + F_2$	$20 = F_7 + F_5 + F_3$
$9 = F_6 + F_2$	$14 = F_7 + F_2$	$23 = F_8 + F_3$
$10 = F_6 + F_3$	$16 = F_7 + F_4$	$26 = F_8 + F_5$
$11 = F_6 + F_4$	$17 = F_7 + F_4 + F_2$	$28 = F_8 + F_5 + F_3$
$12 = F_6 + F_4 + F_2$	$19 = F_7 + F_5 + F_2$	$31 = F_8 + F_6 + F_3$
$13 = F_7$	$21 = F_8$	$34 = F_9$
$21 = F_8$	$33 = F_8 + F_6 + F_4 + F_2$	$54 = F_9 + F_7 + F_5 + F_3$

Table 3

(III) λ -game. Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a given function satisfying $f(1) \leq f(2) \leq f(3) \leq \dots$, $f(n) \geq n$, $n = 1, 2, \dots$. Then one can define (after Schwenk, Fibonacci Quarterly, 1966) the following game played by 2 players. Given a pile of stones, denoted by $(n) \in \mathbb{N} = \{1, 2, \dots\}$. The first player can take i stones from the pile such that $1 \leq i \leq n - 1$. The second player can next take j stones from the pile such that $1 \leq j \leq f(i)$, and the first player can then take k stones such that $1 \leq k \leq f(j)$, and so on. There is no passing. The winner is the one who clears the board.

The case when $f(x) = 2x$ is considered by Gaskel-Whinihan, Fibonacci Quarterly 1963. The winning positions for the second player form a sequence $\{a_1, a_2, \dots\}$ as follows:

$$a_1 = 1, a_{n+1} = a_n + a_j,$$

where j is the smallest integer such that $F_j \geq a_n$.

Example 2. $f(x) = 2x$.

$$\{a_i\} = \{1, 2, 3, 5, 8, 13, \dots\}, \text{ Fibonacci sequence.}$$

Example 3. $f(x) = \sqrt{5}x$.

$$\{a_i\} = \{1, 2, 3, 5, 8, 13, \dots\}, \text{ Fibonacci sequence.}$$

Example 4. $f(x) = 3x$.

$$\{a_i\} = \{1, 2, 3, 4, 6, 8, 11, 13, 15, \dots\},$$

where $a_n = a_{n-1} + a_{n-4}$, $n \geq 6$.

One can show that if the sequence $\{f(n)/n\}$ approaches monotonically (for large n) some real number, then there exists a positive integer R , called the period of the sequence $\{a_n\}$, such that $a_n = a_{n-1} + a_{n-R}$ for sufficiently large n .

The winning strategy for the λ -game is given as follows. Using the sequence $\{a_i\}$, we can express every positive integer in the form

$$n = a_{j_1} + a_{j_2} + \dots + a_{j_r},$$

where $a_{j_1} > f(a_{j_2})$, $a_{j_2} > f(a_{j_3})$, \dots , $a_{j_{r-1}} > f(a_{j_r})$, and the indices j_1, j_2, \dots, j_r are uniquely determined by n . Let us call this decomposition of n Schwenk's f -decomposition of n . If $r = 1$, the pile (n) is a winning position for the second player. If $r > 1$, the pile (n) is a winning position for the first player, who can win by removing a_{j_r} stones from the pile (n). This method is called the "tail-cut".

When $f(n) = \lambda n$, $\lambda > 1$, λ a constant, Furuta has shown that the period $R = R(\lambda)$ satisfies the inequalities

$$\frac{\log \lambda}{\log \lambda - \log(\lambda - 1)} < R(\lambda) \leq \frac{\log(\lambda + 1)}{\log(\lambda + 1) - \log \lambda}.$$

As a corollary, we have $R(\lambda) \sim \lambda \log \lambda$.

As an example, if $f(x) = 3x$, $n = 100$, the Schwenk decomposition of n is $100 = 76 + 21 + 3$. Hence the first player wins by taking 3 stones out of the pile since the tail is 3.

(IV) Game version of Coxeter groups.

Suppose a group G is generated by a set S of involutions (i.e. elements of order 2), $S = \{s_i \mid i \in I\}$. Then we define the length $\ell(\sigma)$ of $\sigma \in G$ with respect to S by

$$\begin{cases} \ell(1) = 0, \\ \sigma \neq 1 : \ell(\sigma) = \min \{r \mid \sigma = s_{j_1} \dots s_{j_r}, s_{j_1}, \dots, s_{j_r} \in S\}. \end{cases}$$

An expression $\sigma = s_{i_1} \dots s_{i_m}$, $s_{i_j} \in S$, $j = 1, \dots, m$, is said to be redundant if $m > \ell(\sigma)$.

The pair (G, S) is called a Coxeter system or Coxeter group with respect to the generators S if every redundant expression

$$\sigma = s_{i_1} \dots s_{i_m}$$

contains 2 indices i_p, i_q such that $p < q$ and

$$\sigma = s_{i_1} \dots s_{i_{p-1}} s_{i_{p+1}} \dots s_{i_{q-1}} s_{i_{q+1}} \dots s_{i_m}.$$

We call this the "2-element deletion law". It is known that G then has the following presentation with respect to S :

$$G = \langle s_1, s_2, \dots \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1, m_{ij} < \infty, i \neq j \rangle.$$

Example 5

$G = S_n$ (the symmetric group of degree n),

$S = \{(1\ 2), (2\ 3), \dots, (n-1\ n)\}$.

If $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix} \in G$, then

$$\ell(\sigma) = \#\{(p, q) \mid 1 \leq p < q \leq n \text{ and } j_p > j_q\}.$$

There is a picturesque way of calculating this number $\ell(\sigma)$. To illustrate, let

$$n = 5, \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}.$$

Imagine 5 persons called Mr 1, Mr 2, Mr 3, Mr 4 and Mr 5, whose heights increase in that order, standing in a row in front of a television set (see Fig. 1). A person will complain about another if the latter is taller and in front of him. Then $\ell(\sigma)$ is equal to the total number of complaints.

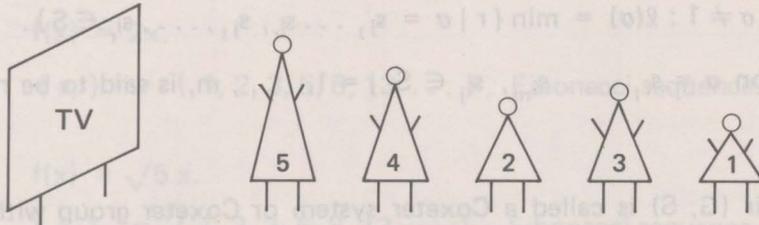


Fig. 1

The number of complaints of the 5 persons are as follows:

Mr 1	.	.	.	4
Mr 2	.	.	.	2
Mr 3	.	.	.	2
Mr 4	.	.	.	1
Mr 5	.	.	.	0
Total	.	.	.	9

Hence $\ell(\sigma) = 9$.

In Example 5, (G, S) is known to be a Coxeter group. This fact has an interesting game version which makes use of the two-element deletion law of Coxeter groups. This game is played by one person. We will give a concrete example corresponding to $n = 5$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}$.

Given 5 vertical segments joined by a finite number of horizontal bridges as shown in Fig. 2. The points upstairs are labelled 1, 2, 3, 4, 5. Starting from each of these points, one moves to a point downstairs according to the following rule. Movement is always vertically downwards until one comes to a bridge, in which case one must cross the bridge. The corresponding destinations for the points upstairs are as shown in Fig. 2. The game is this. Given the configuration of bridges in Fig. 2, what is the maximum number of bridges that can be removed without affecting the destination points?

(V) Hironaka's game

Hironaka has given a game version of the proof of his celebrated desingularisation theorem. M. Spivakoskii has recently given a winning strategy for the case when the characteristic is zero. The winning strategy for the characteristic p case seems to be still unknown. We will only describe the game.

Given a convex set Δ in \mathbb{R}_+^n , where

$$\mathbb{R}_+^n = \{ (x_1, \dots, x_n) \mid x_i \geq 0, i = 1, \dots, n \},$$

such that

- (1) Δ is generated by a finite set $\{v^{(1)}, v^{(2)}, \dots, v^{(k)}\}$, i.e. Δ consists of convex linear combinations of $v^{(1)}, \dots, v^{(k)}$, where the points $v^{(i)}$ have rational coordinates;
- (2) for every $(x_1, \dots, x_n) \in \Delta$, $x_1 + \dots + x_n \geq 1$.

The game is played by 2 players A and B. The first move for player A is to choose a subset $\Gamma^{(1)} \subset \{1, 2, \dots, n\}$ such that

$$(*) \quad \sum_{j \in \Gamma} x_j \geq 1 \text{ for all } (x_1, \dots, x_n) \in \Delta.$$

Player B then chooses an element $i \in \Gamma$ and defines a map $\mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ given by

$$\sigma_{\Gamma, i} : (x_1, \dots, x_n) \rightarrow (x'_1, \dots, x'_n),$$

$$\text{where } x'_i = \sum_{j \in \Gamma} x_j - 1,$$

$$x'_j = x_j, j \neq i,$$

and player B replaces the set Δ by the set $\Delta^{(1)} = \sigma_{\Gamma, i}(\Delta)$. The moves are repeated in this fashion, thereby giving rise to sets $\Gamma^{(1)}, \Delta^{(1)}, \Gamma^{(2)}, \Delta^{(2)}, \dots$.

The winner is determined as follows.

- (1) If there exists p such that there exists $(x_1, \dots, x_n) \in \Delta^{(p)}$ for which $x_1 + \dots + x_n < 1$, then A is the winner.
- (2) If player A cannot find a subset $\Gamma^{(p)} \subset \{1, \dots, n\}$ satisfying property (*), i.e.

$$\sum_{j \in \Gamma^{(p)}} x_j \geq 1 \text{ for all } (x_1, \dots, x_n) \in \Delta^{(p-1)},$$

then B is the winner.