SOLUTION OF PROBLEMS

No problems are proposed in this issue. The solutions to P. 1 – P. 3/80 are as follows.

P. 1/80 Find all real x which satisfy the equation

$$\sqrt{x-2\sqrt{x-1}} + \sqrt{x+3-4\sqrt{x-1}} = 1$$

(Teo Soh Wah)

Solution by K. M. Chan.

The given equation is $\sqrt{(\sqrt{x-1}-1)^2} + \sqrt{(\sqrt{x-1}-2)^2} = 1.$

Let $1 \leqslant x < 2$. The equation becomes $\sqrt{x-1} = 1$, i.e. $x = 2$.

Let $2 \leqslant x \leqslant 5$. The equation becomes an idenity.

Let $x > 5$. The equation becomes $\sqrt{x-1} = 2$. i.e. $x = 5$.

Thus the given equation is satisfied by any x in $2 \leqslant x \leqslant 5$ and by no other x.

P. 2/80 Without obtaining its value, show that the integral $\int_0^{\alpha} \cos(x^2)\, dx$ is positive.

(K. M. Chan)

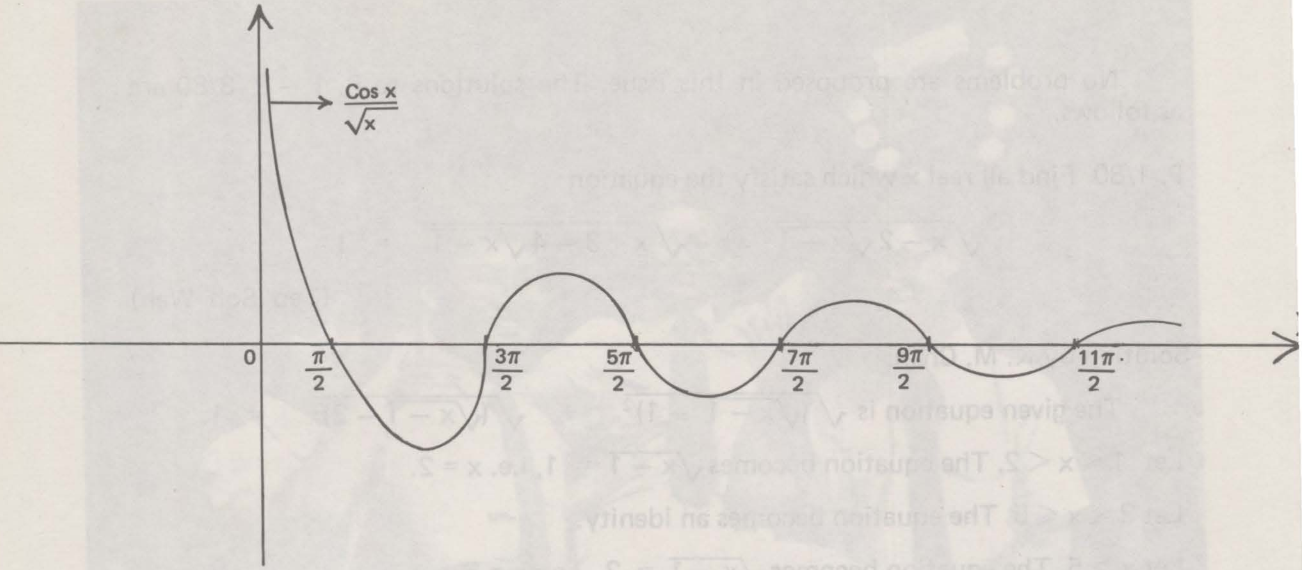
Solution by Proposer

$$\int_0^{\alpha} \cos(x^2)dx = \tfrac{1}{2}\int_0^{\alpha} \frac{\cos}{\sqrt{x}}\, dx$$

$$= \tfrac{1}{2}\left(\int_0^{\frac{3\pi}{2}} \frac{\cos x}{\sqrt{x}}\, dx + \int_{\frac{3\pi}{2}}^{\frac{7\pi}{2}} \frac{\cos x}{\sqrt{x}}\, dx + \int_{\frac{7\pi}{2}}^{\frac{11\pi}{2}} \frac{\cos x}{\sqrt{x}}\, dx + \ldots\ldots\right)$$

$$= \tfrac{1}{2}\ (a_3 + a_7 + a_{11} + \ldots\ldots),\ \text{say.}$$

Now it is obvious that $a_7, a_{11}, \ldots$ etc are positive (see sketch of $\frac{\cos x}{\sqrt{x}}$ below). It is also true that a_3 is positive but this fact is not obvious. Using numerical integration (e.g. by using Maclaurin Series to expand cos x) it will be found that a_3 is approximately 0.6394 (I am indebted to my wife Dr. Y.M. Chow for this computation). This shows that the integral is positive.



P3/80. Simplify, for $x > 1$, each of the following:

(i) $\sqrt{\dfrac{x-1}{x+1}} + \dfrac{2}{(x+1)+\sqrt{x^2-1}}$

(ii) $\arctan \dfrac{x+a+\sqrt{x^2-1}}{\sqrt{1-a^2}} - \arctan \sqrt{\dfrac{(1-a)(x-1)}{(1+a)(x+1)}}$, $|a| < 1$.

(iii) $\left[\dfrac{x+a+\sqrt{x^2-1}-\sqrt{a^2-1}}{x+a+\sqrt{x^2-1}+\sqrt{a^2-1}}\right]\left[\dfrac{\sqrt{(a+1)(x+1)}-\sqrt{(a-1)(x-1)}}{\sqrt{(a+1)(x+1)}+\sqrt{(a-1)(x-1)}}\right]$, $a > 1$,

(iv) $\left[\dfrac{x+a+\sqrt{x^2-1}-\sqrt{a^2-1}}{x+a+\sqrt{x^2-1}+\sqrt{a^2-1}}\right]\left[\dfrac{\sqrt{(a^2-1)(x^2-1)}-(ax+1)}{x+a}\right]$, $a > 1$.

(M.J. Wicks)

Solution by Proposer.

(i) 1

(ii) $\sqrt{\dfrac{1+a}{1-a}}$

(iii) $a - \sqrt{a^2-1}$

(iv) $\sqrt{a^2-1} - a$