NORMAL SUBGROUPS AND FACTOR GROUPS

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In Herstein's book "Topics in Algebra" (2nd Ed.) Problem 1 on p. 53 reads as follows:

If H is a subgroup of a group G such that the product of two right cosets of H in G is again a right coset in G, prove that H is normal in G.

First let me introduce some notation. For any x in G the set

$$Hx = \{ hx \mid h \in H \}$$

is called a right coset of H in G. (The set $xH = \{ xh \mid h \in H \}$ is called a left coset of H in G). Next let me define the product of two subsets of G. If A and B are (non-empty) subsets of G, then their product AB is the set

(1) $$\{ xy \mid x \in A \text{ and } y \in B \} .$$

With these definitions we can restate the problem in the following form:

If H is a subgroup of a group G such that for every pair of elements x, y of G there is an element z of G such that

(2) $$Hx\, Hy = Hz,$$

prove that H is normal in G (i.e. $Hx = xH$ for all x in G).

This is not one of Herstein's starred problems and should therefore be straight forward. My experience is that most students do not find it so. Of course, once a hint is given (such as "make use of the identity of H"), the rest is simple manipulation.

Taking my own hint let me proceed to solve the problem. It follows from (2) that for any $h_1, h_2 \in H$ there is an $h_3 \in H$ such that

$$h_1 x h_2 y = h_3 z.$$

Putting $h_1 = 1$ and $h_2 = 1$ (where 1 denotes the identity of G), we get

$$xy = h_3 z.$$

Therefore

$$HxHy = Hz = H(h_3 z) = Hxy.$$

(That $H(h_3 z) = Hz$ follows from the fact that H is a subgroup of G). Hence we have

(3) $$HxHy = Hxy \quad \text{for all } x, y \text{ in } G.$$

The rest is easy. Putting $y = 1$ in (3), we have

(4) $$HxH = Hx \quad \text{for all } x \text{ in } G.$$

From (4) it follows that for any $h_1, h_2 \in H$ there is an $h_3 \in H$ such that

$$h_1 x h_2 = h_3 x,$$

so that

$$xh_2 = h_1^{-1} h_3 x.$$

Writing $h_4 = h_1^{-1} h_3$, we get

$$xh_2 = h_4 x.$$

This implies $xH \subseteq Hx$. Similarly we can show that $Hx \subset xH$. Hence we have

$$xH = Hx \quad \text{for all } x \text{ in } G.$$

Now in an article in the Mathematical Gazette (Vol. 62, March 1978, No. 419, pp. 29 – 35) I.D. Macdonald asked what can be deduced from (3) above if H is not assumed to be a subgroup of G but just a non-empty subset of G.

First let me explain why Macdonald was interested in this question.

Suppose that H is a (non-empty) normal subset of G (i.e. $Hx = xH$ for all x in G). As in the case when H is a normal subgroup we denote by G/H the set of all right cosets of H in G. We define a binary operation o on G/H by

(5) $$Hx \circ Hy = Hxy.$$

We must point out the o is not the multiplication of two subsets of G as defined by (1).

Now look at the system $\langle G/H, \circ \rangle$. The immediate question that one would like to ask is whether $\langle G/H, \circ \rangle$ is a group. It turns out that $\langle G/H, \circ \rangle$ is always a group. To prove this let us first show that the operation

$$Hx \circ Hy = Hxy$$

is well-defined, i.e. we show that if $Hx_1 = Hx_2$ and $Hy_1 = Hy_2$ then $Hx_1y_1 = Hx_2y_2$. Indeed, we have

$$Hx_1y_1 = Hx_2y_1 = x_2Hy_1 = x_2Hy_2 = Hx_2y_2.$$

The associativity of o is clear. $H = H1$ is the identity and Hx^{-1} is the inverse of Hx.

The above proof is of course valid when H is a normal subgroup of G. There is, however, no talk about equivalence relation or classes. But does $\langle G/H, \circ \rangle$ have the usual properties of the factor group of a normal subgroup? Let us consider an example.

Let G be the symmetric group on the set { 1, 2, 3 } . Then the elements of G (written in the cycle notation) are

1,	(1 2),	(1 2 3),	(1 3 2),	(1 3),	(2 3)
1	a	b	b^2	ab	ab^2

(The second line gives all the elements in terms of a = (1 2) and b = (1 2 3). Let H = { (1 2 3), (1 3 2) } . Then we have

$$H_1 = 1\,H = H1 = \{ (1\,2\,3), (1\,3\,2) \}$$
$$H_2 = (1\,2)\,H = H(1\,2) = \{ (2\,3), (1\,3) \}$$
$$H_3 = (1\,2\,3)\,H = H(1\,2\,3) = \{ (1\,3\,2), 1 \}$$
$$H_4 = (1\,3\,2)\,H = H(1\,3\,2) = \{ 1, (1\,2\,3) \}$$
$$H_5 = (1\,3)\,H = H(1\,3) = \{ (1\,2), (2\,3) \}$$
$$H_6 = (2\,3)\,H = H(2\,3) = \{ (1\,3), (1\,2) \}$$

Thus H is a normal subset of G and $< G/H, o >$ is a group. But $< G/H, o >$ is quite a different group from the usual factor group of a normal subgroup. For one thing, G/H might be expected to have three elements (since G has six elements and H has two), but it has six elements.

We now go back to the question asked by Macdonald. Using the operation o defined by (5) we can phrase the question like this:

Let H be a non-empty subset of a group G. If Hx o Hy = HxHy for all x, y in G, what can be said about H?

After Macdonald asked the question, he continued to say "It would be pleasant if we could say that H is a normal subset of G. This may well be too much to hope for. But actually to produce a group G with a non-empty subset H and an element x_0 such that Hx = HxH for all x but $Hx_0 \neq x_0H$ seems a difficult problem." (Note that Hx o Hy = HxHy means Hxy = HxHy which gives HxH = Hx if we put y = 1).

We now know why the problem is difficult. The group G with the required properties does not exist! What Macdonald had hoped for is actually true and this was proved by B. H. Neumann. (Mathematical Gazette, vol. 62, pp. 298 – 299) I would like to finish this note by presenting Neumann's proof.

We start with the condition that H is a (non-empty) subset such that

(6) $$Hx = HxH \quad \text{for all } x \text{ in } G.$$

Multiplying both sides of (6) on the left by x^{-1}, we get

(7) $$x^{-1}Hx = (x^{-1}Hx)H \quad \text{for all } x \text{ in } G.$$

Multiplying both sides of (6) on the right by x^{-1} and then replacing x^{-1} by x, we get

(8) $$H = H(x^{-1}Hx) \quad \text{for all } x \text{ in } G.$$

Let w be an arbitrary element of $x^{-1}Hx$. Then by (7) we have

$$w = (x^{-1}yx)z \quad \text{for some } y, z \in H.$$

Now

$$w = (x^{-1}yx)z = zz^{-1}(x^{-1}yx)z$$
$$= z(xz)^{-1}y(xz)$$

Since (8) holds for every x in G we have

$$H = H(xz)^{-1}H(xz),$$

so that $w = (x^{-1}yx)z = z(xz)^{-1}y(xz) \in H$.

Hence we have

$$x^{-1}Hx \subseteq H \quad \text{for all } x \text{ in } G.$$

Replacing x by x^{-1}, we get

$$xHx^{-1} \subseteq H \quad \text{for all } x \text{ in } G,$$

or

$$H \subseteq x^{-1}Hx \quad \text{for all } x \text{ in } G.$$

Hence we have $x^{-1}Hx = H$ for all x in G. Therefore H is a normal subset of G.

Let us go back to the beginning. If Herstein had his problem as follows:

If H is a non-empty subset of a group G such that $HxHy = Hxy$ for all x, y in G, prove that H is a normal subset of G.

Then it would probably deserve a two-star rating.