INTER-SCHOOL MATHEMATICAL COMPETITION 1981

PART A

Saturday 4 July 1981 9.00 a.m. – 10.00 a.m.

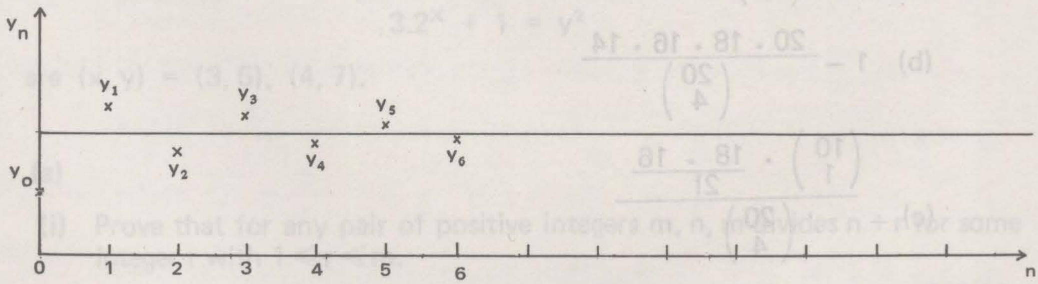
Each question carries 5 marks.

1. Given an isosceles triangle ABC with AB = AC. Let D, E be points on sides BC, AC respectively such that AD = AE. If $\angle BAD = 20^\circ$, find $\angle CDE$.
   (a) 15°
   (b) 10°
   (c) 9°
   (d) 5°
   (e) 20°

2. Evaluate $\int_{-2}^{-1} \frac{dx}{x-1}$
   (a) $\ln 3 - \ln 2$
   (b) $\ln 2 + \ln 3$
   (c) $-\ln(1.5)$
   (d) $\ln(-2) - \ln(-3)$
   (e) none of the above.

3. The number of solutions of the equation $\cos 3\Theta - \sin\Theta = 0$ in the range $0 \leqslant \Theta \leqslant \pi$ is
   (a) 2
   (b) 3
   (c) 4
   (d) 5
   (e) 6

4.



Given an infinite sequence $\{y_n\}$, $y = A + B\,\alpha^n$ as represented in the above sketch. The behaviour of this sequence is indicated by
(a) $0 < \alpha < 1,\ B < 0,\ A > 0,\ A > |B|$
(b) $-1 < \alpha < 0,\ B > 0,\ A < 0,\ |A| < B$
(c) $\alpha > 1,\ B < 0,\ A > 0,\ A > |B|$
(d) $-1 < \alpha < 0,\ B < 0,\ A > 0,\ A > |B|$
(e) $\alpha < -1,\ B > 0,\ A > 0,\ A < B.$

5. The least speed with which a small stone will be thrown in order that it flies infinitely far from the surface of the earth is

(a) 64 km/sec (b) 14 km/sec (c) 12 km/sec (d) 10 km/sec (e) 8 km/sec

(Take $g = 10$ m/sec^2, radius of earth = 6400 km and neglect the effect of the sun).

6. Let f be a function such that for all integers m, n
(i) f(m) is an integer
(ii) $f(2) = 2$
(iii) $f(mn) = f(m)f(n)$
(iv) $f(m) > f(n)$ when $m > n$
The value of f(7) is
(a) 6 (b) 3 (c) 12 (d) 7 (e) 5

7. The number

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{1000{,}000}}$$

lies between
(a) 1995 and 1996
(b) 1996 and 1997
(c) 1997 and 1998
(d) 1998 and 1999
(e) 1999 and 2000

8. Ten pairs of shoes are in a closet. Four shoes are selected at random. Find the probability that there will be at least one pair among the four shoes selected.

(a) $\dfrac{20 \cdot 18 \cdot 16 \cdot 14}{\binom{20}{4}}$

(b) $1 - \dfrac{20 \cdot 18 \cdot 16 \cdot 14}{\binom{20}{4}}$

(c) $\dfrac{\binom{10}{1} \cdot \dfrac{18 \cdot 16}{2!}}{\binom{20}{4}}$

(d) $1 - \dfrac{\dfrac{20 \cdot 18 \cdot 16 \cdot 14}{4!}}{\binom{20}{4}}$

(e) $1 - \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{20 \cdot 19 \cdot 18 \cdot 17}$

9. Given an isosceles triangle ABC with AB = AC. Let circle C_1 be inscribed in Δ ABC. Another circle C_2 is now drawn just touching C_1 (at exactly one point) and sides BA and BC. If BC = 2a and $\measuredangle B = 2\beta$, find the area of C_2.

(a) $\pi a^2 \left(\tan\beta \cot\frac{\beta}{2}(\sec\beta - \tan\beta)\right)^2$

(b) $\pi a^2 \left(\tan\beta \tan\frac{\beta}{2}(\sec\beta - \tan\beta)\right)^2$

(c) $\pi a^2 \left(\tan\beta \cot\frac{\beta}{2}(\sec\beta - \cot\beta)\right)^2$

(d) $\pi a^2 \left(\cot\beta \cot\frac{\beta}{2}(\sec\beta - \tan\beta)\right)^2$

(e) $\pi a^2 \left(\tan\beta \cot\frac{\beta}{2}(\csc\beta - \tan\beta)\right)^2$

10. What is the mean distance between two points taken at random on the circumference of a circle with radius r?

(a) $\frac{r}{\pi}$ (b) $\frac{2r}{\pi}$ (c) $\frac{3r}{\pi}$ (d) $\frac{4r}{\pi}$ (e) $\frac{5r}{\pi}$

PART B

Saturday, 4 July 1981 10.00 a.m. – 12.00 nooon

Attempt as many questions as you can. Each question carries 25 marks.

1. Show that the only solutions in positive integers of the equation

$$3.2^x + 1 = y^2$$

are $(x, y) = (3, 5), (4, 7)$.

2. (a)

(i) Prove that for any pair of positive integers m, n, m divides n + r for some integer r with $1 \leqslant r \leqslant m$.

(ii) Let p and q be prime numbers with $p + 2 = q$ and $p > 3$, prove that 6 divides $\frac{p+q}{2}$.

(b) Let S be an infinite series given as

$$S = 0.7 + 0.097 + 0.00997 + 0.0009997 + 0.000099997 + \ldots .$$

Evaluate S.

3. (a) Let r be a real number greater than one. Suppose that $|f(x) - f(y)| \leqslant |(x - y)|^r$ for all pairs of real numbers x and y.

Show that f (x) is a constant function, i.e. there is a real number c such that f(x) = c for all real numbers x.

(b) Does the conclusion of (a) hold if r = 1? Justify your answer.

4. Let $A = (a_{ij})$ be a n x n matrix (n > 1) such that, for $i \geqslant j$

$$a_{ij} = \frac{a^{i-j} + 2(j-1)}{n-(i-j)} \quad \text{where a is a constant.}$$

If A is symmetric, find an expression for the sum of all the entries of A in terms of n and a and show that if a is an integer, this sum is also an integer.

5. Three circles of equal radius all pass through a point O, and they meet again at three distinct points A, B, and C. Show that the circle ABC has the same radius as the other circles, and that OA is perpendicular to BC.

6. A particle is projected with velocity $\sqrt{2gh}$ under gravity from A so as to pass through P. The horizontal and vertical coordinates of P are a, b ($b<h$) referred to A. By considering the horizontal and vertical motion of the particle, show that the velocity V of the particle at P is given by

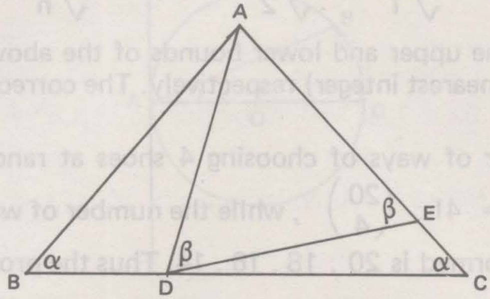
$$V^2 = 2g(h-b).$$

Hence or otherwise show that if the angle between the two paths at P is a right angle, then P lies on the ellipse $x^2 + 2y^2 = 2hy$.

[For the second part of the question, you may use without proof any standard result but they must be stated clearly].

SOLUTIONS TO PART A

1.



Let ∡CDE = x. By the given conditions we have $\beta = \alpha + x$ and $\beta + x = \alpha + 20°$ (see figure above). This gives $x = 10°$. The correct answer is (b).

2. $$\int_{-2}^{-1} \frac{dx}{x-1} = \ln |x-1| \Big|_{-2}^{-1} = \ln 2 - \ln 3$$
$= -\ln(1.5)$. The correct answer is (c).

3. Use graphical method. The curves $\sin\theta$ and $\cos 3\theta$ have exactly three points of intersection in the range $0 \leqslant \theta \leqslant \pi$. The correct answer is (b).

4. Since the given sequence converges to some real number $\ell < \infty$, we have $|\alpha| < 1$. Moreover it alternates about ℓ, thus case (a) is not possible. Finally from $y_0 < y_1$ and the fact that $|\alpha| < 1$, we get $B < 0$. We are left with case (d). Now $A > 0$ and $A > |B|$ imply that $\ell > 0$. The correct answer is indeed (d).

5. Apply equation of motion $\frac{d^2r}{dt^2} = -\frac{k}{r^2}$ . . . . . (1), where $k = \frac{10}{10^3}(6400)^2$ by the given data. Putting $v = \frac{dr}{dt}$, we may write (1) as $v\frac{dv}{dr} = -\frac{k}{r^2}$ . . . . . (2). Let v_0 be the initial velocity. Then at $r = r_0 = 6400$, $v = v_0$. Integrating (2) now yields $\frac{v^2}{2} = \frac{k}{r} + \left(\frac{{v_0}^2}{2} - \frac{k}{r_0}\right)$.

If the particle is to fly infinitely far, we must have $\frac{v^2}{2} - \frac{k}{r_0} \geqslant 0$, or ${v_0}^2 \geqslant \frac{2k}{r_0} = 128$, so $v_0 > 11.5$. The correct answer is therefore (c).

6. Conditions (ii) and (iii) give $f(4) = 4$. From $f(2) < f(3) < f(4)$ we get $f(3) = 3$. Thus $f(6) = 6$ and $f(8) = 8$, which force $f(7) = 7$. The correct answer is (d).

7. We deduce the following inequality easily from the graph of $\frac{1}{\sqrt{x}}$, $x \geqslant 1$:

$$1 + \int_1^n \frac{1}{\sqrt{x}}dx > \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \int_1^{n+1} \frac{1}{\sqrt{x}}\,dx.$$

When $n = 10^6$, the upper and lower bounds of the above inequality are 1999 and 1998 (to the nearest integer) respectively. The correct answer is (d).

8. The total number of ways of choosing 4 shoes at random from 20 shoes is $20 \,.\, 19 \,.\, 18 \,.\, 17 \;=\; 4!\binom{20}{4}$, while the number of ways of choosing 4 such that no pair is formed is $20 \,.\, 18 \,.\, 16 \,.\, 14$. Thus the probability ρ that 4 shoes be selected at random such that no pair is formed is $\dfrac{20 \,.\, 18 \,.\, 16 \,.\, 14}{4!\binom{20}{4}}$, and the required probability is $1 - \rho$. The correct answer is (d).

9.

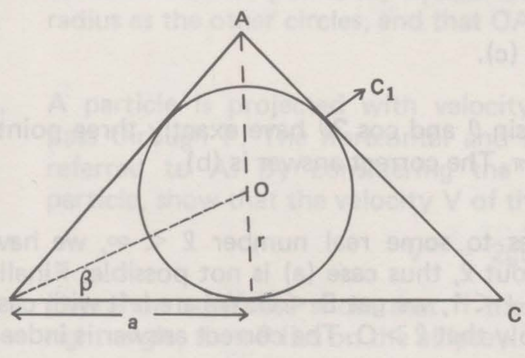


Fig (i)

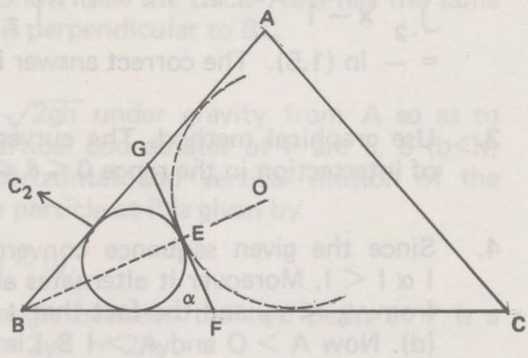


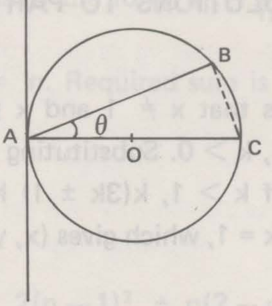
Fig (ii)

Let O be the centre of C_1. Then O lies on the angular bisector of ∡B. We have $r = a \tan\beta$ (See Fig (i)). Thus area of circle $C_1 = \pi\,(a \tan\beta)^2$ . . . . . (1).

Consider Fig (ii). Let FG be the common tangent of C_1 and C_2, where E is the point of contact between the two circles. ΔBFG is isosceles with $BF = BG$ and C_2 is the inscribing circle. Using (1) area of circle C_2 is $\pi\,(EF \tan\frac{\alpha}{2})^2$, where $\alpha = \frac{\pi}{2} - \beta$. Thus area required is $\pi(EF)^2\,(\tan(\frac{\pi}{4} - \frac{\beta}{2}))^2$. Now $EF = BE \tan\beta$, where $BE = BO - r = a \sec\beta - a \tan\beta$. Area of C_2 is therefore $\pi a^2\,(\tan(\frac{\pi}{4} - \frac{\beta}{2}))^2 \tan^2\frac{\beta}{2}\,(\sec\beta - \tan\beta)^2$.

None of the given answers is correct.

10.



Let A be any point on the circumference of a given circle with radius r. Further, let B be the second random point on the circumference with ∡BAC = θ, where $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ (see figure above, taking AOC to be the initial line). We have AB = 2r cos θ. The mean distance between A and B is thus

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r\cos\theta \, d\theta = \frac{4r}{\pi}$$

The correct answer is (d).

SOLUTIONS TO PART B

1. Direct substitution shows that $x \neq 1$ and $x \neq 2$. Thus $x \geqslant 3$. Clearly $2 \nmid y$ and $3 \nmid y$, so $y = 6k \pm 1, k > 0$. Substituting in $3.2^x + 1 = y^2$ yields $2^{x-2} = k(3k \pm 1)$ . . . . . (1). If $k > 1$, $k(3k \pm 1)$ has odd divisor greater than 1, contradicting (1). Hence $k = 1$, which gives $(x, y) = (3, 5)$ or $(4, 7)$.

2.(a) (i) Let s be the remainder of $n + m$ on division by m. Then $m \mid (n + m) - s$. Hence $m \mid n + r$, where $r = m - s$ and $1 \leqslant r \leqslant m$.

(ii) From (i) we see that $3 \mid p, p + 1$ or $p + 2$. Since p, q are primes with $p > 3$ and $q = p + 2$, we get $3 \mid p + 1$, where $p + 1 = \frac{p+q}{2}$. Moreover p is odd. Thus $2 \mid \frac{p+q}{2}$ and $6 \mid \frac{p+q}{2}$.

(b) Let $U = 0.3 + 0.003 + 0.00003 + \ldots$. Then $U + S = 1 + 0.1 + 0.01 + 0.001 + \ldots$ We have $U + S = \frac{10}{9}$ and $U = \frac{30}{99}$. Hence $S = \frac{80}{99}$.

3.(a) We have $| f(x) - f(y) | \leqslant | x - y | | x - y |^{r-1}$, so $| \frac{f(x) - f(y)}{x - y} | \leqslant | x - y |^{r-1}$ Then $\lim_{y \to x} | \frac{f(x) - f(y)}{x - y} | \leqslant \lim_{y \to x} | x - y |^{r-1} = 0$, where $r > 1$. So $f'(x) = 0$ for each y, which implies that f(x) is a constant function.

(b) The conclusion is false if $r = 1$. For example, let $f(x) = x$. Then $| f(x) - f(y) | = | x - y | \leqslant | x - y |$. But f(x) is not a constant.

4. For $i > j$, we consider the terms for which $i - j = k$, $k = 1, 2, \ldots, n - 1$. Set $S_k = \sum_{i-j=k} a_{ij}$, where $a_{ij} = \frac{a^k}{n-k} + (\frac{2}{n-k})(j-1)$, for $j = 1, 2, \ldots, n - k$. S_k is an A.P. We get $S_k = a^k + n - 1 - k$. And

$$\sum_{k=1}^{n-1} S_k = (\sum_{k=1}^{n-1} a^k) + \sum_{k=1}^{n-1}(n-1) - \sum_{k=1}^{n-1} k$$

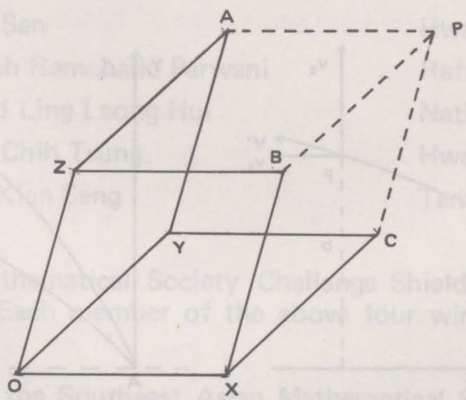
$$= \frac{a(1 - a^{n-1})}{1 - a} + (n-1)^2 - \frac{(n-1)n}{2}$$

For $i = j$, there are n terms of the form $a_{ii} = \frac{1}{n}(1 + 2(j-1))$, $j = 1, 2, \ldots,$ n. This gives $\sum_{j=1}^{n} a_{ij} = n$. Required sum is therefore, by symmetry of A, equal to $n + 2 \sum_{k=1}^{n-1} S_k$.

$$= \frac{2a(1-a^{n-1})}{1-a} + 2(n-1)^2 + n(2-n)$$

Since $\frac{1-a^{n-1}}{1-a}$ is a polynomial in a, if a is an integer, this sum is also an integer.

5. Let X, Y, Z be the centres of the three circles. The circles with centres Y and Z meet at O and A, so OYAZ is a rhombus. Similarly OZBX and OXCY are rhombi. It is easy to show using vectors or properties of parallelograms that the lines through A, B, C parallel to OX, OY, OZ respectively meet at a point P, and that BXCP etc are rhombi: all the lines in the figure have equal length. Hence P is the centre of the circle ABC, and the radus of this circle is PA = XO. Finally $OA \perp ZY \parallel BC$, as may easily be proved.



6. Let u_1 and v_1 be the horizontal components of the velocities of the particle at A and P respectively, while u_2 and $_2$ the respective vertical components see Fig (i). Then $v_1 = u_1$ and $v_2^2 = u_2^2 - 2gb$. Thus $V^2 = v_1^2 + v_2^2 = (u_1^2 + u_2^2) - 2gb = 2gh - 2gb = 2g(h - b)$.

Taking the origin of co-ordinates at P, we get $y = x \tan \alpha - \dfrac{gx^2}{2V^2}(1 + \tan^2 \alpha)$ (See Fig (ii)).

The path passes through A = (a, –b), so

$$-b = a \tan \alpha - \frac{ga^2}{2V^2}(1 + \tan^2 \alpha)$$

i.e. $ga^2 \tan^2 \alpha - 2aV^2 \tan \alpha + (ga^2 - 2V^2 b) = 0$ . . . . (1)

If the two possible paths at P are at right angle, then $(\tan\alpha_1)(\tan\alpha_2) = -1$, where $\tan\alpha_1$ and $\tan\alpha_2$ are the two roots of the quadratic equation (1). Thus $-1 = \dfrac{ga^2 - 2V^2 b}{ga^2}$, or $2hb = a^2 + 2b^2$, using $V^2 = 2g(h - b)$. Thus the point P(a, b) with respect to the axes AX, AY lies on the curve

$$X^2 + 2Y^2 = 2hY.$$

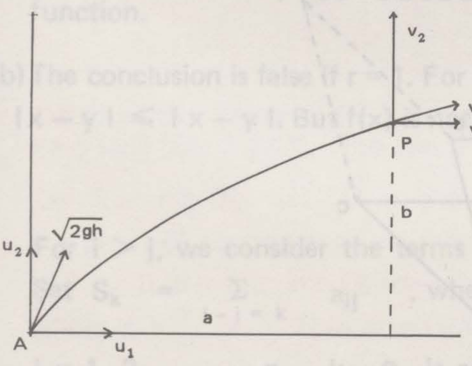


Fig (i)

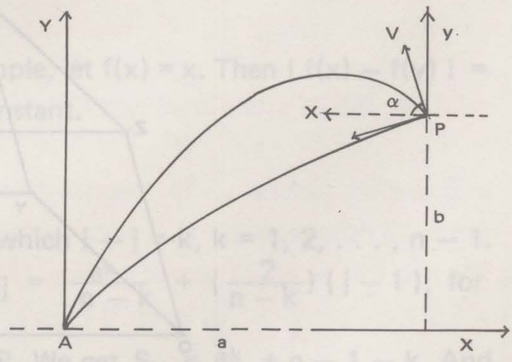


Fig (ii)