EVOLUTION OF THE NOTION OF MAGNITUDE*

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Magnitudes and Numbers

The concepts of counting, length, area, volume, weight, etc. occur in every society and seem to be an inherent part of human experience. These primitive notions are referred to as magnitudes and they are used primarily for purposes of making comparisons (order relation) and for combining quantities (operations). It should be noted that each of the above examples of magnitude is as primitive as any of the others. The idea of arranging them into some kind of hierarchy of primitiveness occured very late and only for mathematical convenience.

Numbers and numerals were the mathematical objects and symbols introduced to represent magnitudes. The choice of a unit (e.g., sheep, mile, square, feet, kg, etc.) serves to indicate the type of magnitude being represented.

Pre-Greek Period (? – 600 B.C.)

The primary concern during this period was with practical problems hence the emphasis was on the operations, i.e. effective systems of notation, computations and algorithms. The Babylonians were capable of approximating any magnitude by rational numbers to any desired degree of accuracy. Questions involving concepts and meaning did not seem to have been considered at all.

Greek Period (600 B.C. – 300 A.D.)

The emphasis was on concepts – clarity and avoidance of contradictions – hence on axioms and on definitions. With the exception of Diaphantos mathematicians of note ignored computational problems.

The systems of numbers and of magnitudes developed mostly by Eudoxos (∼ 350 – 400 B.C.) as described by Euclid are as follows.

1. **Pure Numbers – (Rationals)**

   (a) A "number" – a positive integer >1 – is taken as *primitive* and axiomatized,

   i.e.

   $$\mathbb{N} = \{2, 3, 4, \ldots\}$$

   with order relation $\leqslant$ and operations $+$, $\cdot$ are primitives subject to well known axioms.

*Lecture delivered at the National University of Singapore – February 1981.

(b) A "rational" $\frac{p}{q}$ (for $p, q \in \mathbb{N}$) is *defined* as an operator : $nq \to mp$ for every $n \in \mathbb{N}$.

With the order relations

$$\frac{p}{q} \lesseqgtr \frac{p'}{q'} \quad \text{iff} \quad pq' \lesseqgtr p'q$$

and only one operation is allowed, i.e. composition of operators, which is equivalent to multiplication:

$$\frac{p}{q} \cdot \frac{p'}{q'} = \frac{pp'}{qq'}$$

2. Magnitudes

(a) The notion of magnitude is axiomatized : magnitudes of a given type are the elements of a set M endowed with an order relation ($\leqslant$) and *one* binary operation (+) only subject to the well known axioms plus the so called Archimedean property (used centuries before Archimedes), i.e.

(Axiom 1) For every $A, B \in M$ there exists a $k \in \mathbb{N}$ such that $kA > B$. | Note: $kA = A + A + \ldots + A$ (k terms) |.

(b) Ratios or Proportions

In order to relate magnitudes to pure numbers, Eudoxos introduced the *definitions* of ratio of two magnitudes of the same type.

Definitions. For any $A, B \in M$, $\frac{A}{B}$ is the operator: $kB \to kA$, for every $k \in \mathbb{N}$.

$$\frac{A}{B} \cdot \frac{A'}{B'} = \frac{A}{B} \circ \frac{A'}{B'} \quad \text{(composition of operators)}$$

$$\frac{p}{q} \lesseqgtr \frac{A}{B} \quad \text{iff} \quad pB \lesseqgtr qA \quad \text{for } p, q \in \mathbb{N}$$

and

(**) $\frac{A}{B} \leqslant \frac{A'}{B'}$ iff for every $p, q \in \mathbb{N}$: $\frac{p}{q} \leqslant \frac{A}{B} \Rightarrow \frac{p}{q} \leqslant \frac{A'}{B'}$

Then one more axiom:

(Axiom 2) For every $A, A', B' \in M$ there exists a $B \in M$ such that $\frac{A}{A'} = \frac{B}{B'}$

| Note: if $B' = 1$ were permitted, this would give a 1 – 1 correspondence between M and the set of ratios $\{ \frac{A}{B} \; ; \; A, B \in M \}$ |.

The number 1 was very deliberately not considered a number. It was reserved to represent a unit for a given type of magnitude. The above system avoided the problem of comparing different types of magnitudes and of having to give meaning to expressions such as (2 sheep) x (3 sheep) in sets of sheep, especially since product of two lengths was supposed to yield area and not length.

Middle Ages Period (300 A.D. – 1600 A.D.)

The Emphasis was again on operational properties of numbers – development of algebra and algorithms for solving algebraic equations.

Zero and negative numbers appeared in Hindu mathematics in late middle ages, also imaginary numbers were used by Italian algebraists (Tartaglia, Ferrari, Cardano, Bombelli) in the 16th century. However these came up purely formally in commputations as expressions of the form: $2 - 3$, $\sqrt{5-7}$, etc. They were called false, absurd, imaginary, impossible, fictitious, etc.

Modern Period (1600 – 1900)

1579 – R. Bombelli noted that given a unit of length there is a 1 – 1 correspondence between lengths and ratio of lengths. He thus introduced directly +,. on set of lengths and represented numbers by lengths thereby obtaining a geometric representation of $\mathbb{R}$ (idea attributed to Descartes). He also gave modern definitions of negative numbers and of complex numbers (attributed to Gauss and Argand). (He may not have been widely read as Book IV of his "Algebra" was published only recently).

1591 – F. Viéte introduced modern algebraic notation, in particular letters for numbers appearing as coefficients in polynomials; symbols +, –; a quadrature A^2 for A, etc., thus opening up consideration of general problems.

1634 – S. Stevin introduced similar views as Bombelli. He took a number to be an "essentially continuous" magnitude (without making it very precise). He introduced *decimal* 'fractions' in modern notation and has an effective modern algorithm for approximating any "number" by them. Actually, he concentrated on finding roots of a polynomial equation $P(x) = Q(x)$ with $\deg P > \deg Q$ and $P(O) < Q(O)$. He anticipated Bolzano's ideas by trying $x = 10, 10^2, \ldots$ until $P(10^k) > Q(10^k)$ for the first time. Then followed by $x = 10^{k-1} + j$ with $j = 1, 2, \ldots, 9$, etc.

1637 – R. Descartes showed how to represent A.B and A/B as lengths for any given lengths A and B, thereby allowing +,. to be defined on the set of lengths. These were taken to represent numbers. He introduced notations A^3, A^4, . . . $\sqrt{A}$, etc. (AA was still written for A^2). (Fermat was more explicit in formulating similar views of lengths and coordinates).

1664 - 1666 – I. Barrow in his lectures at Cambridge returned to Eudoxos and defended his motivation. He defined number as a symbol denoting a ratio of magnitudes then with the operations +,., he obtained $\mathbb{R}$. His ideas were essentially taken up by Newton, Dedekind, Cantor with little change.

1812 – F. Gauss in his paper on Hypergeometric Series introduced sup., inf., lim sup, lim inf.

1821 – A. Cauchy considered numbers as magnitudes with +,. and subject to the Cauchy Criterion of completeness (which he assumed was true, Bolzaus at least tried to prove this property).

1880 – Weierstrass detached the notion of number from that of length and defined IR as the completion of the set of rationals. (similar work has been done by Meray – Cantor)
– Dedekind returned to Eudoxos and introduced "Dedekind cuts" of rational numbers as the definition of a real number.
– Dedekind and Cantor developed the theory of real numbers.

Reference

N. Bourbaki, Elements of Mathematics.
Book III. General Topology, Part 1 (1966) Ch. IV Real numbers – Historical Note pp. 406 – 416.