THE INTRODUCTION OF COMPLEX NUMBERS*

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Any keen mathematics student will tell you that complex numbers come in when you want to solve a quadratic equation $ax^2 + bx + c = 0$ when $b^2 < 4ac$. However, if one tries to find out how they first came into mathematics then the surprising answer is that they were first introduced in the process of solving not quadratic but cubic equations. When Bombelli (1526–1572) first discovered what we call complex numbers he wrote (in translation):

"I have found a new kind of tied cube root very different from the others."

By a "tied cube root" he means a cube root expression like $\sqrt[3]{(2 \div \sqrt{-121})}$ where there is a square root under a cube root.

How did he come to consider such monstrous expressions? And why weren't complex numbers introduced in the context of solving quadratic equations as they are today? It is the purpose of this paper to try to answer these questions.

2. The Greeks

It is often said that the Babylonians (in the second millennium B.C.) and the Greeks (no later than about 300 B.C.) knew how to solve quadratic equations. In fact, if you look at what they wrote then there are no traces of x's, no ideas of polynomial equations of degree 2, 3, etc. What we do find are problems of the form: A rectangle has two adjacent sides whose total length is 10 and its area is 24. What are the lengths of the sides?

We think of this as $a + b = 10$, $ab = 24$ and then move to the equation $x^2 - 10x + 24 = 0$, regarding a, b as the roots and using the fact that the sum of the roots is 10 and the product 24. But this was not the Babylonian or Greek way. The Babylonians gave a recipe (formula is not quite the right word in its modern sense), the Greeks a geometrical construction, to give the answers. The Babylonians never considered (so far as we knew) cases where problems had complex solutions. The Greeks could never have produced a complex solution because their constructions produced actual lines and you cannot draw a line of complex length (even though we do now use Argand diagrams for representing complex numbers).

3. al-Khwarizmi

From at least the seventh century A.D. Hindu mathematicians treated (the equivalent of) quadratic equations and they explicitly said that negative quantities do not have square roots. Much of their mathematics was transmitted to the Arabs but, curiously, the Arabs did not, so far as we know, use negative quantities.

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al-Khwarizmi (9th century A.D.), from whose name we get the word "algorism" or "algorithm", wrote the first book on algebra – indeed he called his book *Hisāb al-jabr w'almuqābala* (830 A.D.) and that is why we use the Anglicized form of al-jabr even today.

This book classifies equations and shows how to solve them. Since al-Khwarizmi did not use negative numbers he classified quadratic equations in the following sorts of way:

Square equal to numbers,
Square plus roots equal to numbers,
Squares equal to numbers plus roots, etc.

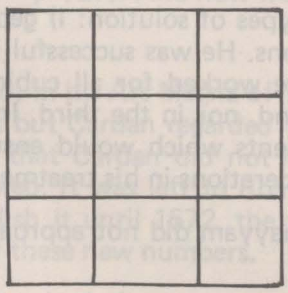
In our notation (with b, c positive):

$x^2 = c$,
$x^2 + bx = c$,
$x^2 = c + bx$.

He gave two kinds of solution. I shall call these geometrical and radical. Consider the equation

$$x^2 + 10x = 39$$

and consider the diagram



where the central square has side x and each oblong adjoining it has its other side 2½. Then the figure without the corner squares has area $x^2 + 4(2\frac{1}{2}x) = x^2 + 10x$. If we add the corners, each of area $(2\frac{1}{2})^2$, we get the whole area to be

$$x^2 + 10x + 4(2\tfrac{1}{2})^2 = x^2 + 10x + 25.$$

But $x^2 + 10x = 39$ so $x^2 + 10x + 25 = 39 + 25 = 64$. Thus the area of the whole figure is 64 and, since it is a square, its side is 8. Hence $x = 8 - (2\frac{1}{2} + 2\frac{1}{2}) = 3$. This is the geometrical solution.

The radical solution is that given by (essentially) the well-known formula for solving a quadratic. Paraphrasing al-Khwarizmi we have "square half the coefficient of x [$(10/2)^2$] and add the numbers [+39]. Total 64. Take its root, 8. Subtract half the coefficient of x [10/2]. Answer 3."

He neglected the other (in this case negative) root.

4. Omar Khayyam

Three hundred years later algebra had advanced considerably and cubics were being treated. It would appear that the work of Diophantos (who probably lived some time between 150 and 350 A.D.) had been rediscovered. Diophantos was sophisticated enough to consider not only cubes but also fourth, fifth, sixth powers. Just how much Diophantos influenced the Arabs we do not know, but Omar Khayyam (more famous for his Rubaiyat) wrote a book in which he gave a fine treatment of cubics.

The major breakthrough came from a theorem of Archimedes (The Sphere and the Cylinder proposition II.5). This gave a *geometric* method for solving a cubic equation. Omar Khayyam used this technique and said that if one wanted to solve a cubic equation then conics must come in (and not just ruler and compass). He did not justify this remark except practically. He solved cubics using properties of conics. What this amounts to is producing lines which solve cubic equations. Then, of course, if a numerical answer is required, the line must be measured.

Omar Khayyam's (not completely fulfilled) aim was: a) to classify cubics and b) to solve them all. He classified them in the same way as al-Khwarizmi, considering the various types such as

cubes equal to squares, roots and numbers, ($x^3 = ax^2 + bx + c$ with a, b, c all positive.)

He wished to give three types of solution: i) geometrical solutions, ii) radical solutions and iii) integer solutions. He was successful in the first endeavour, for the geometric constructions he gave worked for all cubics (with real coefficients). He was not succcessful in the second, nor in the third. In the latter case he wished to find conditions on the coefficients which would ensure an integral solution. Diophantos employed similar considerations in his treatment of equations.

Not surprisingly, Omar Khayyam did not approach complex numbers for they had no place in the geometry.

5. The Italians

By about 1200 Arab culture was becoming better known in Europe. Fibonacci, otherwise known as Leonardo of Pisa, went across to North Africa where his father worked in the customs. There he learned of Hindu-Arabic numerals (0, 1, 2, . . . , 9) and he is generally regarded as one of the first to introduce these into Europe. He travelled a lot and learned a great deal of mathematics from the Arabs and in one of his works he did solve a cubic equation. He even showed that his equation $10x + 2x^2 + x^3 = 20$ did not have an integral nor a rational solution, and he worked out an approximate answer to a high degree of accuracy (1.368808). All this he did by Euclidean geometry, not using conics. Indeed, Omar Khayyam's work did not seem to have become known for a very long time.

From Fibonacci onward there were a lot of Italians working on algebra, but it was not until the end of the fifteenth century that great strides were made.

Luca Pacioli wrote a book in 1494 – the first printed book on algebra as opposed to arithmetic and in this book he spent a lot of time doing manipulations with square roots and cube roots. However, Pacioli was of the opinion that one could not solve cubic equations by radicals. (In a sense he was right: one cannot solve all cubic equations using only real roots even if one restricts the coefficients to be real.)

Shortly afterwards, probably around 1510 or slightly later, Scipio dal Ferro did solve cubics. Whether he knew how to solve all types is unclear. Basically his treatment was the modern one. After removing any x^2 term by an appropriate substitution (in $x^3 + ax^2 + bx + c = 0$ put $y = x + a/3$) one is left with either $x^3 = px + q$ or $x^3 + px = q$ where p, q are both non-negative. Neglecting the easy case of p or $q = 0$, in essence, Scipio's treatment was to compare the equation with $x^3 = (u^3 + v^3) + 3uvx$ which he obtained from $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, where $x = u + v$.

The problem then is to find numbers u, v such that $u^3 + v^3 = q$ and $3uv = p$. If we write $a = u^3$, $b = v^3$ then the problem is to find numbers a, b such that $a + b = q$ and $ab = p^3/27$. But this problem is one whose solution was known to the Greeks, as we noted near the beginning of this paper. Having found a, b we have

$$x = \sqrt[3]{a} + \sqrt[3]{b}$$

where $a, b = +q/2 \pm \sqrt{((q/2)^2 - p^3/27)}$. And now we see where Bombelli's tied cube roots' come in.

In fact Cardan published, in his Ars Magna of 1545, the solution of a quadratic equation with complex roots but Cardan regarded these as sophistic and useless. It appears from other writings that Cardan did not have any clear grasp of complex numbers and how they worked. It was left to Bombelli, who wrote his Algebra in the 1550's but did not publish it until 1572, the year of his death, to give a full, formal and clear treatment of these new numbers.

Bombelli was adept at manipulating expressions involving radicals. Presumably he employed the same rules for his tied cube roots and also performed calculations such as in (in our notation) $(2 + i)^4 = 2 + 11i$. In treating one cubic equation he came up with the expression $\sqrt[3]{(2 + \sqrt{0-121})} + \sqrt[3]{(2 - \sqrt{0-121})}$ from which he obtained $(2 + \sqrt{0-1}) + (2 - \sqrt{0-1})$ which equalled 4. What he then did was to substitute this solution back into his cubic and it worked!

Bombelli had also been suspicious of these new numbers but having employed them for a while, he came to accept them and overcame his misgivings. For him the proof of the pudding seems to be in the eating!

Even though Bombelli gave rules such as (in modern, but not very different, notation)

$$(+i).(+i) = -,$$
$$(+i).(-i) = +$$

it was still quite a long time before complex numbers were totally accepted. The final justification of the formula for the solution of the cubic did not come until 1686 when Leibniz showed by substituting the formal solution back in the cubic that the formula always gave a solution. By that time the use of letters for variables became common practice and this allowed a general treatment which was not possible a century or so before.

6. Conclusion

Thus we see that the geometric context of problems which we regard as polynomial equations militated against the introduction of complex numbers for a very long time, and it was not until a more 'algebraic' approach was adopted and the solution of cubic equations was given a recipe or 'formula', that it transpired that numbers formally defined did lead to solutions — even when, in the intermediate stages, those numbers were imaginary. Even then it was a long time before these new numbers were formally justified — first by Leibniz in the sense that they did give proper solutions of the cubic and later, in the nineteenth century, by Hamilton when he reduced complex numbers to pairs of real numbers with specially defined operations for addition and multiplication — but that is beyond the scope of our present essay.