INTER-SCHOOL MATHEMATICAL COMPETITION 1980

PART A

Saturday, 31 May 1980 9.00 a.m. – 10.00 a.m.

Each question carries 5 marks.

1. The square ABCD has sides of length 2 cm. The centre of the square is M and A′, B′, C′, D′ are the mid-points of AM, BM, CM, DM, respectively. The area of the part common to the two parallelograms AB′CD′ and BC′DA′ is equal to
   (a) $3/2$;
   (b) $3\sqrt{2}/2$;
   (c) $2\sqrt{2}$;
   (d) $\sqrt{2}$;
   (e) none of the above.

2. The numbers $a_1, a_2, a_3, \ldots$ form a geometric progression with $a_1 = 1$, $a_2 = r$ and $r \neq 1$. The sequence $b_1, b_2, b_3, \ldots$ is obtained as follows:

$$b_1 = a_1, b_2 = a_2 a_3, b_3 = a_4 a_5 a_6 \ ;$$

and inductively, if for i, λ we have

$$b_i = a_\lambda a_{\lambda+1} \ldots a_{\lambda+i-1} \ ,$$

then

$$b_{i+1} = a_{\lambda+i} a_{\lambda+i+1} \ldots a_{\lambda+2i} \, .$$

Then $b_n = r^k$, where k is equal to
   (a) $\frac{1}{2}n(n-1)$;
   (b) $\frac{1}{2}n(n-1)(n+1)$;
   (c) $\frac{1}{2}n(n+1)(2n+1)$;
   (d) $\frac{1}{2}n(n+1)(n+2)$;
   (e) none of the above.

3. Among all the triangles with given base and given perimeter, the triangle with maximum area is necessarily
   (a) right-angled;
   (b) obtuse;
   (c) equilateral;
   (d) isosceles;
   (e) none of the above.

4. The subnormal of a plane curve at a point P is (by definition) the segment PN, where N is the point of intersection of the normal at P and the x-axis. A curve whose subnormal at every point has length 1 is necessarily
   (a) a circle;
   (b) a parabola;
   (c) an ellipse;
   (d) a hyperbola;
   (e) none of the above

5. The regular pentagon $P_1P_2 \ldots P_5$ is inscribed in a circle of unit radius. Circles $C_1, C_2, \ldots, C_5$, of equal radii, are such that C_i has centre P_i and touches (externally) the pair of circles with centres at the neighbouring vertices. The area of that part of the interior of the pentagon which is bounded by successive arcs of the circles $C_1, C_2, \ldots, C_5$, respectively, is
   (a) $5(1 - 5\alpha \tan 2\alpha) \tan 2\alpha$;
   (b) $5(\sin 2\alpha - 2\alpha \cos 2\alpha) \cos 2\alpha$;
   (c) $5(\sin 2\alpha - 3\alpha \cos 2\alpha) \cos 2\alpha$;
   (d) $5(\cos 2\alpha - 3\alpha \sin 2\alpha) \sin 2\alpha$;
   (e) $5(\sin 2\alpha - 5\alpha \cos 2\alpha) \sin 2\alpha$.

   Here, α is such that $10\alpha = \pi$.

6. Five circles divide the plane into a number of disjoint regions bounded by arcs of the circles. The maximum number of closed regions that can be obtained in this way is
   (a) 29;
   (b) 30;
   (c) 31;
   (d) 32;
   (e) none of the above.

7. A cube of edge 6 cm. is divided into $6^3 = 216$ unit cubes by planes parallel to the faces of the cube. A sphere of diameter 6 cm. is inscribed in the large cube. The number of (complete) unit cubes contained in the sphere is
   (a) 48;
   (b) 56;
   (c) 64;
   (d) 72;
   (e) none of the above.

8. Let f be given by

$$f(x) = \left(\frac{\sin x}{x}\right)^{x^{-2}}, x \neq 0.$$

   Then $\lim_{x \to 0} f(x)$
   (a) is 1 ;
   (b) is $\frac{1}{4}\pi$;

(c) is $\ln 6$;

(d) is $e^{-\frac{1}{6}}$

(e) does not exist.

9. In a sequence of independent trials, the probability of success in a trial is constant and equal to p, $0 < p < 1$. Let α_n be the probability of obtaining n successes in a run of 2n trials; and let β_n be the probability for n + 1 successes in a run of 2n + 1 trials, $n \geqslant 1$. If G, L are the sets defined by

$$G = \{ n \mid \alpha_n > \beta_n \}, \quad L = \{ n \mid \alpha_n < \beta_n \},$$

then

(a) G is infinite if $p < 2/3$;
(b) L is infinite if $p < 2/3$;
(c) L is finite if $p < 2/3$;
(d) G is non-empty if $p \geqslant 2/3$;
(e) L is finite if $p \geqslant 2/3$.

10. A uniform rod AB rests in limiting equlibrium with A on a horizontal floor and B in contact with a vertical wall. The coefficients of friction at each point of contact are the same and all forces acting on the system are in the same vertical plane. If a mass is suspended from a point P on AB, then
(a) the rod will slip for any position of P;
(b) the system remains in equilibrium for any position of P;
(c) the system remains in equilibrium only if P is the mid-point of AB;
(d) the system remains in equilibrium only if P is below the mid-point of AB;
(e) the rod will slip if P is above the mid-point of AB.

PART B

Saturday, 31 May 1980 10.00 a.m. – 12.00 noon

Attempt as many questions as you can.

Each question carries 25 marks.

1. Let p, q be whole numbers, neither of which is a multiple of 3. Show that one of $p - q$ or $p + q$ is a multiple of 3.
If further, neither p nor q is a multiple of 5, show that 15 is a factor of $p^4 - q^4$.

2. The sequence $F_1, F_2, F_3, \ldots$ is such that

$$F_{n+2} = F_{n+1} + F_n, \; n \geqslant 1,$$

and $F_1 = F_2 = 1$. Prove that

(i) For all $n \geqslant 6$, $F_n \geqslant (\sqrt{2})^n$.

(ii) For each k and for all $n \geqslant k+2$,

$$F_n = F_{k+1}\,F_{n-k} + F_k\,F_{n-k-1}.$$

Deduce that 5 is a factor of F_{5k} for each $k \geqslant 1$. Find another number p such that p is a factor of F_{pk} for each $k \geqslant 1$.

3. Find the integral

$$\int \frac{dx}{(x+a)\sqrt{(x^2-1)}},\ x > 1,$$

in each of the following cases:

(i) $|a| < 1$;

(ii) $a = 1$;

(iii) $a > 1$.

4. (a) The function f is defined on the integers and takes real values which satisfy

$$f(m+n) = f(m)\ f(n),$$

for every pair of integers m, n. If f is not identically zero, show that $f(0) = 1$ and $f(1) \neq 0$. Find an expression for $f(n)$ in terms of n and $f(1)$.

(b) The function g is defined on the real numbers and takes real values which satisfy

$$g(x+y) = g(x)\ g(y),$$

for every pair of real numbers x,y.

Prove that if g is differentiable at $x = 0$, then g is differentiable at each real number x, and that

$$g'(x) = g'(0)\ g(x),$$

where $g'(x)$ is the differential coefficients of g at x.

5. Let s be the sum of the lengths of the edges of a rectangular parallelopiped; A the sum of the areas of its faces; and V the volume. Prove that

$$144\ V^{\frac{2}{3}} \leqslant 24\ A \leqslant s^2.$$

Find the conditions for which

$$144\ V^{\frac{2}{3}} = 24\ A = s^2.$$

6. A spherical shell of centre O has a perfectly reflecting inner surface. The points P_0, P_1 are on the surface and a ray of light is emitted from P_0 towards P_1.

Prove that the reflected ray will pass through P_0 again if and only if there is a rational number λ such that

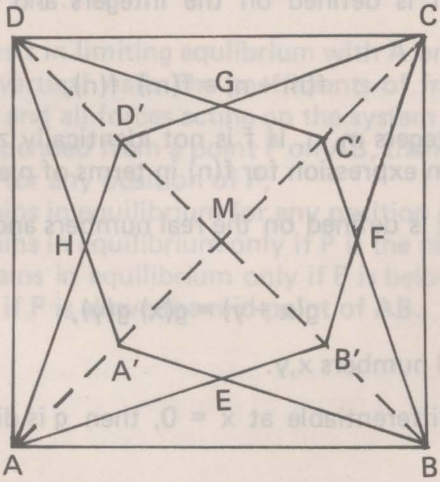
$$P_0\widehat{P_1}0 = \lambda\pi \text{ and } 0 < \lambda < ½.$$

Deduce that a ray emitted from any point P inside the sphere, and directed towards P_1, is such that its path and that of its reflections will be periodic if and only if $\widehat{PP_1}0$ is a rational multiple of π.

[You may assume the law of reflection: the reflected ray is in the plane determined by the incident ray and the normal (at the point of incidence) and the normal bisects the angle between these two rays.]

SOLUTIONS TO PART A

1.



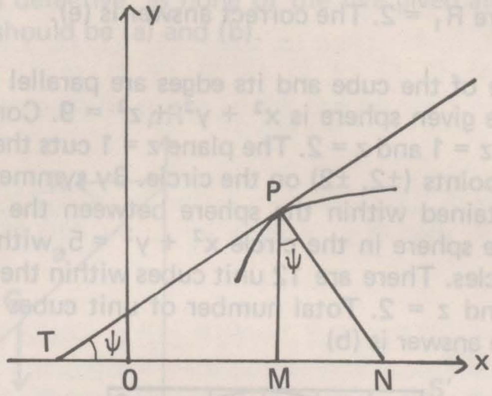
Let E, F, G, H be the points as shown, so that what is required is the area of the octagon A′EB′FC′GD′H.

Let S be the area of the quadrilateral A′EB′M. Then by symmetry area required is 4S. Consider $\triangle$ABM, where E is the centroid. Then S can be computed from the following relation: 2 x Area AB′M − S + Area ABE = Area ABM. The correct answer is (e).

2. The number of "factors" (i.e. the number of a's) in the product which gives b_i is i. Hence the total number of a's used to get $b_1, b_2, \ldots, b_{n-1}$ is $\frac{1}{2}n(n-1)$. Let $b_n = a_\lambda a_{\lambda+1} \ldots a_{\lambda+n-1} \ldots$ (1). Then $\lambda = \frac{1}{2}n(n-1)+1$, so $a_\lambda = r^{1/2n(n-1)}$. Thus if $b_n = r^k$, we obtain by equating the powers of r on both sides of (1) that $k = \frac{1}{2}n(n-1)(n+1)$. The correct answer is (b).

3. Let AB be the given base and P the vertex of one of the triangles with given perimeter. It follows that AP + PB is constant, and hence, that P lies on an ellipse whose foci are A and B. The maximum area of ΔABP occurs when P is at greatest distance from AB. By symmetry this occurs when P is on the perpendicular bisector of AB, i.e. when ΔABP is isoceles. Alternatively, one can solve by using calculus. The correct answer is (d).

4. It is clear that a circle, with centre on the x-axis, or a straight line parallel to the x-axis, satisfies the given condition.



Let P = (x, y) and let the tangent and normal at P meet the x-axis at T and N, respectively. Then (assuming the curve is smooth),

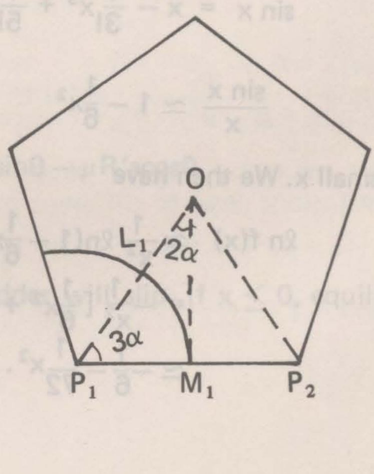
$$\frac{dy}{dx} = \tan\psi \ \ldots (1) \text{ and } y = PN\cos\psi \ \ldots (2).$$

From (1) and (2) and given that PN = 1, we get

$$y^2 = [\, 1 + (\frac{dy}{dx})^2 \,] = 1.$$

The case $\frac{dy}{dx} = 0$ gives straight lines $y = \pm 1$. The case $\frac{dy}{dx} \neq 0$ gives circles with centres on the x-axis. We may regard straight lines as circles with infinite radii. Hence the correct answer is (a).

5. Let M_1 be the mid-point of P_1P_2, O is the centre of the circle C, and L_1 the point of intersection of OP_1 and the circle C_1. The required area is 10 times the area bounded by the segments OL_1, OM_1 and the arc L_1M_1 of C_1. The correct answer is (d).

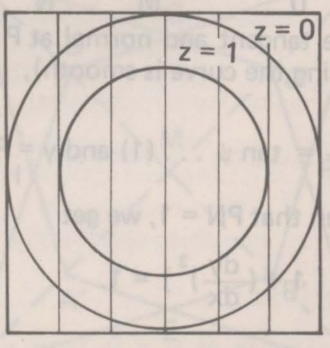


6. Suppose (n−1) circles have already been placed in the plane such that R_{n-1} regions have resulted. An nth circle is now added. Then there are at most 2(n−1) points of intersection of this new circle with the (n−1) circles. These 2(n−1) points divide the circumference of the nth circle into at most 2(n−1) disjoint arcs. Thus the number of additional regions that the nth circle creates is at most 2(n−1). This proves that

$$R_n \leqslant R_{n-1} + 2(n-1).$$

Thus $R_5 \leqslant 22$, where $R_1 = 2$. The correct answer is (e).

7. Let O be the centre of the cube and its edges are parallel to the axes O_x, O_y, O_z. Equation of the given sphere is $x^2 + y^2 + z^2 = 9$. Consider plane sections at the levels z = 0, z = 1 and z = 2. The plane z = 1 cuts the sphere in the circle $x^2 + y^2 = 8$, with points (±2, ±2) on the circle. By symmetry, there are 2.16 = 32 unit cubes contained within the sphere between the planes z = ±1. The plane z = 2 cuts the sphere in the circle $x^2 + y^2 = 5$, with points (±2, ±1) and (±1, ±2) on the circles. There are 12 unit cubes within the sphere and between the planes z = 1 and z = 2. Total number of unit cubes within the sphere is 32 + 2.12 = 56. The answer is (b)



8. A "plausible" solution can be obtained by using power series to obtain approximations. The power series for sin x is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots,$$

so

$$\frac{\sin x}{x} \simeq 1 - \frac{1}{6}x^2$$

for small x. We then have

$$\ell n\, f(x) \simeq \frac{1}{x^2}\,\ell n(1 - \frac{1}{6}x^2)$$

$$= -\frac{1}{x^2}\,[\,\frac{1}{6}x^2 + \frac{1}{2}(\frac{1}{6}x^2)^2 + \frac{1}{3}(\frac{1}{6}x^2)^3 + \ldots]$$

$$\simeq -\frac{1}{6} - \frac{1}{72}x^2.$$

Thus, as $x \to 0$, we have $\ell n\ f(x) \to -\frac{1}{6}$, and hence

$$\lim_{x \to 0} f(x) = e^{-\frac{1}{6}}$$

The correct answer is (d).

9. This question is defective, as none of the five given alternatives is correct. The correct answer should be (a) and (b).

10.

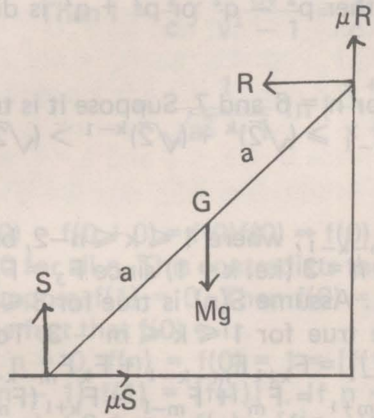


Fig. 1

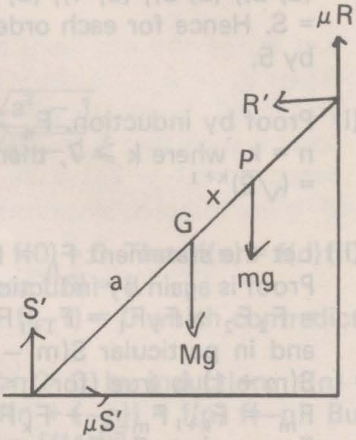


Fig. 2

Refer to Fig. 1. $R = \mu S$ . . . . (1)
Taking moments about G, we have

$$Sa\cos\theta - \mu Sa\sin\theta - Ra\sin\theta - \mu Ra\cos\theta = 0 \ . \ . \ . \ (2)$$

From (1), (2), we get

$$\tan\theta = \frac{1-\mu^2}{2\mu} \ . \ . \ . \ . \ . \ (3)$$

Refer to Fig. 2. $R' = \mu S'$

The clockwise moment about G is

$$\begin{aligned} & mgx\cos\theta + S'a\cos\theta - \mu S'a\sin\theta - R'a\sin\theta - \mu R'a\cos\theta \\ = \ & mgx\cos\theta + S'a\cos\theta\,(1 - \mu^2 - 2\mu\tan\theta) \\ = \ & mgx\cos\theta\,, \end{aligned}$$

in view of (3). Hence if $x > 0$, the ladder will slip. If $x \leq 0$, equilibrium is maintained. The correct answer is (e).

SOLUTIONS TO PART B

1.(i) Every integer can be expressed in the form 3n, 3n + 1 or 3n + 2, where n is an integer. Now both p and q are of the form 3n + 1 or 3n + 2. We want to show that one of p − q or p + q is of the form 3n. Let $p, q \in \{1, 2\}$. Then $(p, q) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\} = S$. Thus for each ordered pair $(p, q) \in S$, one of p − q or p + q is divisible by 3.

(ii) Now $p^4 - p^4 = (p^2 - q^2)(p^2 + q^2)$. By (i) $p^2 - q^2$ is divisible by 3. As (3, 5) = 1, we have to show that one of $p^2 - q^2$ or $p^2 + q^2$ is divisible by 5. As in (i) let $p, q \in \{1, 2, 3, 4\}$. Then $(p, q) \in \{(1, 1,), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\} = S$. Hence for each ordered pair (p, q) either $p^2 - q^2$ or $p^2 + q^2$ is divisible by 5.

2(i) Proof by induction. $F_n \geqslant (\sqrt{2})^n$ is true for n = 6 and 7. Suppose it is true for n = k, where $k \geqslant 7$, then $F_{k+1} = F_k + F_{k-1} \geqslant (\sqrt{2})^k + (\sqrt{2})^{k-1} > (\sqrt{2})^{k-1}2 = (\sqrt{2})^{k+1}$

(ii) Let the statement $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$, where $1 \leqslant k \leqslant n-2$, be S(n). Proof is again by induction. S(n) is true for n = 3 (i.e. k = 1) since $F_3 = F_2 + F_1 = F_2F_2 + F_1F_1 = F_{1+1}F_{3-1} + F_1F_{3-1-1}$. Assume S(n) is true for $3 \leqslant n \leqslant m$ and in particular S(m − 1) and S(m) are true for $1 \leqslant k \leqslant m - 3$. To show S(m + 1) is true (for $1 \leqslant k \leqslant m - 1$). $F_{m-1} = F_{k+1}F_{m-k-1} + F_kF_{m-k-2}$ and $F_m = F_{k+1}F_{m-k} + F_kF_{m-k-1}$. Hence $F_{m+1} = F_m + F_{m-1} = F_{k+1}(F_{m-k} + F_{m-k-1}) + F_k(F_{m-k-1} - F_{m-k-2}) = F_{k+1}F_{(m+1)-k} + F_kF_{(m+1)-k-1}$ for $1 \leqslant k \leqslant m - 3$. To complete the proof of S(m + 1) we have to show the result holds for k = m − 2 and m − 1. $F_{m+1} = F_m + F_{m-1} = F_{(m-1)+1}F_2 + F_{m-1}F_1$ (i.e. k = m − 1). Next $F_{m+1} = F_m + F_{m-1} = 2F_{m-1} + F_{m-2} = F_3F_{m-1} + F_2F_{m-2}$ (i.e. k = m − 2).
Note $F_5 = 5$. Put n = 5m and k = 5 in (ii) and we have $F_{5m} = F_6F_{5(m-1)}) + 5F_{5m-6}$. Assume 5 is a factor of $F_{5(m-1)})$, then by induction 5 is a factor of F_{5m} for all m.
Since 12 is a factor of $F_{12} = 144$ and $F_{12m} = F_{13}F_{12(m-1)} + 144F_{12m+3}$ it follows that 12 is a factor of F_{12m} for all m.

3. $$I = \int \frac{dx}{(x + a)\sqrt{x^2 - 1}}, x > 1.$$

There are many ways of simplifying the integrand by substitution. Different substitutions lead to different expressions for I. We choose the substitution $x = \frac{1}{2}(u + u^{-1})$ so that $x^2 - 1 = [\frac{1}{2}(u - u^{-1})]^2$ and $dx = \frac{1}{2}(1 - u^{-2})\,du$

Then $$I = 2\int \frac{du}{(u + a)^2 + 1 - a^2}$$

(i) $|a| < 1$. Let $b^2 = 1 - a^2$, $b > 0$ and $u + a = bv$

$$\text{Then } I = \frac{2}{b}\int \frac{dv}{1+v^2} = \frac{2}{\sqrt{1-a^2}} \arctan \frac{u+a}{\sqrt{1-a^2}}$$

$$= \frac{2}{\sqrt{1-a^2}} \arctan \left[\frac{x+a+\sqrt{x^2-1}}{\sqrt{1-a^2}}\right].$$

(ii) $a = 1$. $I = 2\int \frac{du}{(u+1)^2} = \frac{-2}{u+1} = \frac{-2}{x+\sqrt{x^2-1}+1}$.

(iii) $a > 1$. Let $c^2 = a^2 - 1$, $c > 0$ and $u + a = cv$.

$$\text{Then } I = \frac{2}{c}\int \frac{dv}{v^2-1} = \frac{1}{c} \ln\left(\frac{u+a-c}{u+a+c}\right)$$

$$= \frac{1}{\sqrt{a^2-1}} \ln \left[\frac{x+\sqrt{x^2-1}+a-\sqrt{a^2-1}}{x+\sqrt{x^2-1}+a+\sqrt{a^2-1}}\right]$$

4(a) $f(0) = f(0+0) = f(0)f(0) \Rightarrow f(0) = 0$ or 1. Suppose $f(0) = 0$. Then $f(n) = f(n)f(0) = 0$ for all n. This contradicts the assumption. Hence $f(0) = 1$.
Suppose $f(1) = 0$. Then $f(0) = f(1+(-1)) = f(1)\, f(-1) = 0$ which contradicts the fact that $f(0) = 1$.
If $n = 0$, $f(n) = f(0) = 1 = [f(1)]^{\circ} = [f(1)]^n$. If $n > 0$, by induction, $f(n) = f(n-1)\, f(1) = [f(1)]^n$. If $n < 0$, $1 = f(0) = f(n + (-n)) = f(n)\, f(-n)$. But $f(-n) \neq 0$, hence $f(n) = [f(-n)]^{-1} = \{[f(1)]^{-n}\}^{-1} = [f(1)]^n$.

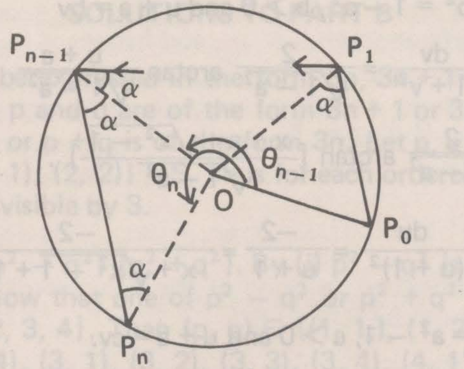
(b) $g(0) = 0$ or 1. If $g(0) = 0 \Rightarrow g(x) = 0 \Rightarrow g'(x) = 0 = g'(0)g(x)$. If $g(x) \neq 0$ for all x, then $g(0) = 1$.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{g(x)g(h)-g(x)}{h}$$

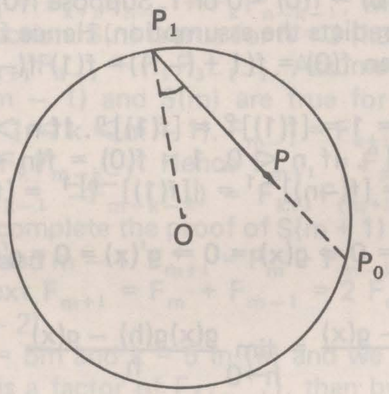
$$= \lim_{h\to 0} g(x) \frac{[g(h)-g(0)]}{h} = g'(0)g(x).$$

5. Let a, b, c be the lengths of the edges. Then $S = 4(a+b+c)$, $A = 2(ab+bc+ca)$ and $V = abc$. Hence $S^2 = 8[(a-b)^2 + (b-c)^2 + (c-a)^2 + 6(ab+bc+ca)] \geqslant 48(ab+bc+ca) = 24A$. Next, $(a^2b^2c^2)^{1/3} \leqslant \frac{1}{3}(ab+bc+ca)$ or $144V^{2/3} \leqslant 24A$. If $a = b = c$, we have equality.

6.



Let P_n be the nth point of incidence of the ray of light and let $P_0\hat{O}P_n = \theta_n$ Then $\theta_n - \theta_{n-1} + 2\alpha = \pi(\text{mod } 2\pi)$, where $P_0\hat{P}_1 0 = \alpha$. So $\theta_n = n(\pi - 2\alpha)$ (mod 2π). Light ray will pass through P_0 if and only if there is a positive integer n such that $\theta_n = 0(\text{mod } 2\pi)$. That is $n(\pi - 2\alpha) = 2k\pi$, where k is an integer, or $n(1 - 2\lambda) = 2k$, where $\alpha = \lambda\pi$. Hence $0 < \lambda < \frac{1}{2}$ and $\lambda = \frac{p}{q}$ where p, q are integers, since $n(q - 2p) = 2pk$ has solutions $n = 2q$ and $k = q - 2p$.



Let P_1P cut surface at P_0. The reflected ray will follow the same path as that of a ray emitted from P_0. Hence the path of the reflected ray emitted from P will be periodic if and only if that of the reflected ray emitted from P_0 will be periodic, i.e. $P_0\hat{P}_1 0$ is a rational multiple of π (from above), i.e., $P\hat{P}_1 0$ is a rational multiple of π.