GEOMETRY AND PHYSICS*

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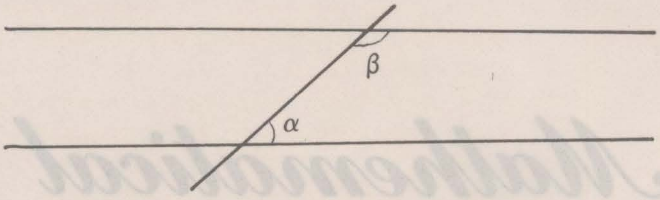
Euclid

Geometry studies the properties of our space. Physics studies physical phenomena. As the latter takes place in space, geometry forms the natural basis of a physical theory. By experience our space is of dimension three; in analytical geometry we describe a point by three coordinates. By its very nature geometry is simpler than physics. It was possible to establish the subject on a set of simple axioms and to deduce the properties of the space through logical conclusions. The great achievement of Euclid has profoundly affected the thinking of mankind. For centuries Euclid was not only a book on geometry but also a book on logic. On close scrutiny, however, not all the Euclidean axioms are simple or evident. The classical example is the Fifth Postulate which says that if two lines are cut by a transversal and if the



angles α and β have a sum less than 180°, the two lines will intersect when extended in these directions. An important consequence of this axiom is the beautiful theorem that the sum of the angles of any triangle is 180°. Efforts to prove this axiom as a logical consequence of the other axioms resulted in failure, a happy failure which led to the creation of the non-euclidean geometries, by J. Bolyai, Gauss, and N.I. Lobacevskii. The discovery of the non-euclidean geometries was one of the great human intellectual triumphs.

In spite of the success of Euclid it is not clear why our space should be Euclidean. The great French mathematician Henri Poincaré held the view that Euclidean geometry surpasses other geometries based on axiomatics by its "simplicity". Even if the latter is accepted as a criterion, what is simple to one may not be simple to others. A serious effort to solve the space problem was made by Hermann von Helmholtz in 1868, which was to rely on the axiom of free mobility in space[1]. This led to the Euclidean and non-Euclidean spaces. Except for Poincaré's view I see no other reason why our space is Euclidean and not non-Euclidean. It is remarkable, and indeed mysterious, that the application of axiomatic Euclidean geometry to daily problems has not led to a contradiction or an absurdity. This should be the strongest argument supporting Euclidean geometry.

* Public lecture delivered at the University of Singapore on 27 June, 1980. This work was done under partial support of NSF grant MCS 77-23579. I wish to dedicate this paper to Professor Yung-Chow Wong, both as a testimony of a long friendship and as an expression of my admiration of the impressive work he has done in promoting the contact of mathematicians of the countries in Southeast Asia.

Geometry and Physics

Can the choice of geometry be decided by physics? A natural process is to relate the motions of the three-dimensional space with the motions of rigid bodies. In practise, however, the form of a body is affected by temperature, external force, etc., and is not rigid. This led Einstein [2] to the conclusion that physical laws are a combination of geometry and physics, in symbols (G) + (P), and physics does not determine (G). If modifications are necessary, he would change (P), but not (G). He went on to say: "If mathematical theorems are related to reality, they are not certain and if they are certain, they are unrelated to reality."

In any case Euclidean geometry fits well with Newtonian mechanics.

Groups of Transformations

In the 18th and 19th centuries projective geometry flourished. Projective properties, based on projections and sections, do not involve metrical concepts. There were also other geometries, such as line geometry in space, circle and sphere geometries, etc. This prompted Felix Klein to formulate in 1870 his Erlangen Program which aimed at a unification of geometry. Klein defined geometry as the study of the invariants of a space under a group of transformations. Euclidean and non-Euclidean geometries fall under this program by the Cayley-Klein model. As a result the same analytical conclusion can often be interpreted to give entirely different geometrical theorems.

Among the geometries in the sense of Klein is Minkowski's geometry which plays a fundamental role in the special theory of relativity. A four-dimensional Minkowski space is the space (x, y, z, t) provided with the quadratic form

$$Q = x^2 + y^2 + z^2 - t^2 .$$

Minkowski geometry is the study of the invariant properties of the space under the group of linear homogeneous transformations on the coordinates x, y, z, t, which leave Q invariant. The group has also a geometrical interpretation in sphere geometry. It is essentially the group of all contact transformations in space which map spheres into spheres and planes into planes. The corresponding geometry is called Laguerre's sphere geometry.

Riemannian Geometry

A most profound achievement in geometry was Riemann's creation of Riemannian geometry in 1854. In providing a theoretical foundation of geodesy Gauss published in 1827 a paper entitled "General investigations of curve and surfaces". This should be considered the birth certificate of differential geometry; before that differential geometry was a branch of the infinitesimal calculus concerned with its geometrical applications. Gauss showed that metrical geometry on a surface can be developed locally, i.e., in the neighbourhood of a point. Riemann developed the Riemannian geometry for a high-dimensional abstract manifold, thus giving a grandiose generalization of Gauss's work. Riemann's paper was his "Habilitationsvortrag". Gauss was in the audience and was deeply impressed.

Relativity

It was remarkable that these two important developments in geometry had their respective effects on physics: Minkowski geometry on the special theory of relativity and Riemannian geometry on the general theory of relativity. When Einstein developed his general relativity theory, the geometrical tools he needed were in existence. Through the study of electricity and magnetism it was found that time is not separate from space: there is only a four-dimensional space-time. The Minkowski space supplied the model. The space-time in the general theory of relativity has a fundamental quadratic differential form which is of signature +++–. It is called a Lorentzian space and is a generalization of the Minkowski space, just as a Riemannian space is a generalization of the Euclidean space.

There are many Lorentzian spaces,depending on the choice of the quadratic differential form, and Einstein's problem was to find one which represents the physical space-time. These spaces satisfy the Einstein equations. In order to express them one needs the powerful tool of tensor analysis, a mathematical formalism adapted to the treatment of geometry in arbitrary coordinates.

Unified Field Theory

The special theory of relativity deals with electricity and magnetism while the general theory of relativity deals with gravitation. This separate status was unsatisfactory and Einstein tried in his later years, without great success, to build a unified field theory which would cover both electricity and magnetism and gravitation. Nowadays the grand design in physics is to have a unified field theory which covers also weak and strong interactions.

An immediate idea for a unified field theory is to extend the basic geometric structure. Thus the five-dimensional space (Kaluza-Klein) and the projective geometry of paths (Veblen) came into play. Einstein tried many structures, among which are: a non-symmetric g_{ik}, complex and hermitian geometry, and general metric spaces. None of the results is conclusive.

Weyl's abelian gauge field theory [3]

It turns out that in all the efforts at a unified field theory the most fecund idea was supplied by Hermann Weyl in 1918. Weyl proposed a gauge field theory in which he introduced a gauge potential to account for electricity and magnetism. Einstein expressed his admiration at the depth and boldness of Weyl's proposal, but gave a number of criticisms. The criticisms were valid, but could be removed by using a "phase potential", instead of a gauge potential.

In the terminology of modern differential geometry Maxwell's theory of electricity and magnetism is a connection in a circle bundle over a four-dimensional Lorentzian manifold. The geometrical object is therefore a family of circles parametrized by the manifold. The important difference from Weyl's original theory is that they are circles, and not lines. (Gauge field theory is a misnomer; it should more correctly be called a phase field theory.) The first set of Maxwell's equations expresses the strength of the field in terms of the curvature of the connection. The second set of Maxwell's equations has also a simple geometric meaning.

An experiment proposed by Y. Aharanov and D. Bohm in 1959 and performed by R.G. Chambers in 1960 gives an electric-magnetic field in a non-simply-connected domain whose strength is zero, but whose phase factor can be observed.

A circle bundle is locally a product. A fundamental mathematical question is whether it is globally a product. This is a sophisticated problem whose solution depends on developments in algebraic topology and differential geometry. It is now possible to describe all the distinct circle bundles over a given manifold and to give the "invariant" (called a characteristic class) which will distinguish the circle bundles.

Dirac knew (1931) the existence of non-trivial circle bundles, i.e., bundles which are globally not products. He identified them with the magnetic monopoles; they are physically not yet identified. The corresponding invariant, an integer, is called a geometric quantum number. On the possible existence of a magnetic monopole Dirac said: "No change whatever in the formalism . . . Under these circumstances one would be surprised if Nature had made no use of it."

Vector Bundles

A circle bundle (resp. line bundle) is the one-dimensional special case of a sphere bundle (resp. vector bundle). In general a vector bundle is a family of vector of vector spaces (called fibers) parametrized by a manifold such that the linear structure on the fibers has a meaning. This is done by using the general linear group GL(q; R), q = dim of fiber, to match the fibers. If the group is O(q), the unit vectors on the fibers have a meaning and we have a sphere bundle. When the fibers are complex vector spaces, we use the groups GL(q; C), U(q), or SU(q). The abelian gauge field theory of H. Weyl discussed above is concerned with a U(1)-bundle.

A vector bundle is locally a product, but may not be so globally. The simplest invariants to measure this deviation are called the characteristic classes. When the bundle is given a connection, the characteristic classes can be expressed in terms of the curvature of the connection. There is thus a close relationship between characteristic classes and local properties.

The first non-abelian gauge field theory in physics was introduced in 1954 by Yang-Mills in their study of the isotopic spin. Mathematically it makes use of an SU(2)-bundle with a connection; the restriction on the connection is given by the Yang-Mills field equations. The successful unified field theory of electricity and magnetism and weak interactions by Salam and Weinberg (1967) uses the non-abelian gauge field theory. It is widely recognized that gauge field theory will play a fundamental role in future developments of theoretical physics.

In 1975 Yang remarked to me: "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere". Actually the development of the concepts of bundles and connections in mathematics had a long history, till it reached its present form [4]. In fact, if we deal with differentiable manifolds, then bundles are in abundance. The tangent bundle of a manifold is in a sense a linear approximation of the manifold. When the manifold lies in an ambient space, all the normal vectors form its normal bundle. In the simplest case of a surface in the three-dimensional Euclidean space, all the unit tangent vectors to the surface form the circle bundle most easily visualized. Its study is highly interesting and provides the clue to many more general problems. In the 30's the "Hopf circle fibering" $S^3 \to S^2$ excited the algebraic topologists, because it was the first example of a continuous mapping of a space to one of lower dimension which is not homotopic to a constant mapping. These examples, and others, show that bundles occur in mathematics from different contexts.

Why Gauge Theory

In recent years by attending physics conferences I often heard discussed the question "why gauge theory"? I wish to give a justification from the mathematical viewpoint.

The first notion of a function can be expressed as a map

$$R \longrightarrow R.$$

This is generalized to a map

$$R^m \longrightarrow R^q,$$

that is, a vector-valued function over R^m. If the domain R^m is generalized to a manifold M, we have a vector-valued function over M:

$$M \longrightarrow R^q,$$

which in turn can be written

$$M \longrightarrow M \times R^q,$$

the mapping to the first factor being the identity. Conceptually this generalization is to replace the Euclidean space R^m by the manifold M which is only locally Euclidean. Manifolds play an important role in Einstein's general theory of relativity.

Some generalization should be made on the range $M \times R^q$, and a natural step is to replace it by a general vector bundle E. Then the generalization of a vector-valued function on M is a "section"

$$M \longrightarrow E,$$

which assigns to a point of M a point on its fiber. Sections generalize vector fields or tensor fields and form an important object in mathematics. It must be more than coincidence that this generalization is also important for the applications of mathematics to physics.

References

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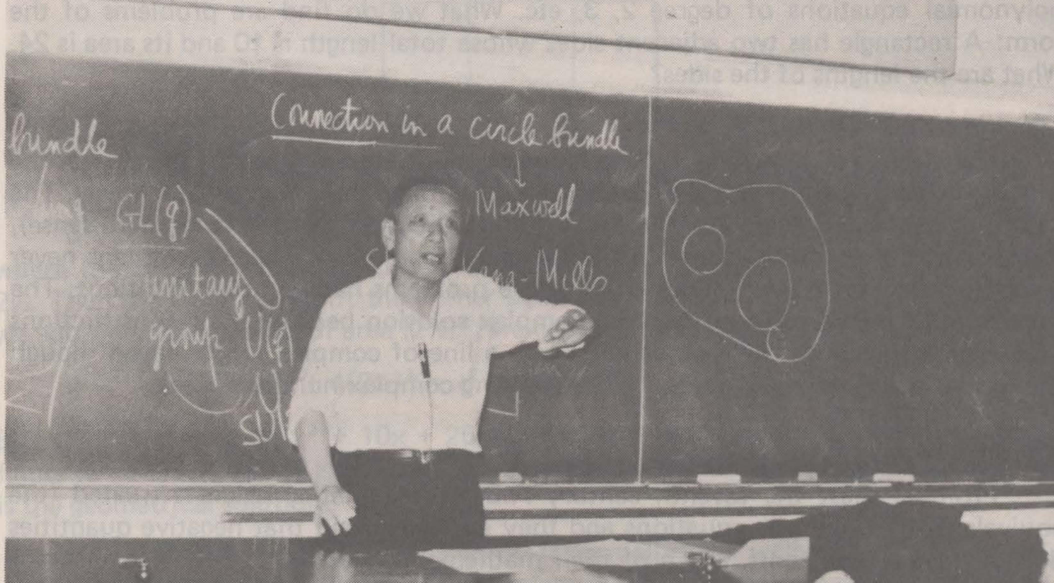
4. Cf., for instance, S. Chern, *Complex manifolds without potential theory*, Second edition, Springer 1979.



Professor S. Chern delivering the Public Lecture on "Geometry and Physics."