Part II: Discussion

Professor Sen. Ladies and gentlemen, I call the forum to order. Well, you have heard from the panel members. Professor Tan told you that changes should be slow and teacher training should be done so that it is adapted to the new syllabus. They have spoken to you about the public examinations, the level of teaching at which you should give what, and that mathematicians are not the only people who should express their views but that you should take the views of the others in developing a mathematics curriculum. Mr Thiagarajah has given us a picture of the performance of Singapore students under syllabuses B and C, and he tells that it is not yet possible to say which syllabus is superior to the other as far as the performance of students indicate. Professor Wong has given us a picture of the Hong Kong scene as far as mathematics education is concerned, and I think we in Singapore will have to go through exactly the same type of process because we are changing from the British system to the Singapore system in our schools. Professor Bullen has said that we must serve the purpose of the second group of mathematicians and he has also touched upon teachers' education in Canada. Professor Schwartz expresses the point of view that secondary school teachers should be taught by university professors, and in that case we can maintain the standards of teachers' education. Professor Dieudonné has raised so many points that I could not take them down. Now I open the forum to the public for questions.

Question. I do not understand what is meant by saying we cannot think non-commutatively.

Professor Dieudonné. When I say "non-commutatively", I should have also said "non-linearly". "Non-commutatively" even more. Except for special cases, it is very hard to get an intuition into non-commutative groups. Commutative groups are very much easier to visualise, be they finite or finitely generated. But non-commutative things are very hard to work with. And if you want an example of linearisation, it is precisely the non-commutative case. You see, when you work with the non-commutative groups, it is very difficult and then the German mathematician Frobenius had a very bright idea, which looks foolish at first: try to imbed a group into a linear group. Why should linear groups be chosen as the kind of model for groups? It turns out to be a fantastically powerful method of studying non-commutative groups and I think the idea is powerful because the general linear group is a group of transformations of vector spaces which you can certainly handle much better. For instance, consider a simple group of even order. Now there is an old theorem of Cauchy which says that in a group of even order, there is an element u which is what we call an "involution", i.e. $u^2 = 1$. Now this does not seem to be something which is very powerful in a group. But it is. Why is that? Because it is easy to visualise an element of a linear group which has that property. The vector space then splits into a direct sum $V \oplus V'$ in which the mapping u is the identity on V and an asymmetery on V'. So we can handle involutions in a linear group by visualising them. And that is a very powerful way. It is just an example, of course. There are better ideas than that. It just shows you how useful it is to imbed a group, about which you cannot say much at all, into a general linear group, about which you already have geometric intuition.

It is thus emphasized that geometric intuition is fundamental for a mathematician. Geometry is to be understood in the widest possible sense, not just 2, 3 or n dimensions. It should be allowed in all sorts of things which have no direct connection with geometry. It is not just the geometric intuition of the three-dimensional world which we live in, but the transfer of the geometric language. When we deal with objects which behave like planes, left translations, rotations and so forth, even if it is over a finite field or infinite-dimensional spaces or even modules over finitary rings, the geometric intuition involved in thinking of things as geometric objects is extremely useful.

Professor Sen. You have told us what the role of education should be for the first group. For the second group, the professional mathematicians, what do you think is the role of creativity?

Professor Dieudonné. You see, I do not like the word "creativity" because it is one of those words whose meaning has been modified. It is very bad to have inflation and inflation of words is bad. Creativity is something which should be restricted to the real thing. When you say that Gauss was a very creative man, I would agree. Out of nothing, he invented mathematics. You see, Einstein was a creative genius. I entirely agree with that. He created relativity out of nowhere. That is fundamental. That is what I call "creativity". I get letters or read papers by people teaching in schools saying you must enhance the creativity of school children. What does "creativity of schoolchildren" mean? Nothing at all. You see, even in the 3rd or 4th year of university, what they do is to learn the tools of the trade first. Well, how do you do that? There are a lot of problems which are somewhat more or less artificial but some are not artificial at all, and then solving them give you a lot of ideas how to work with the tools of mathematics. The things are not obvious in general. You have to use them. You do not talk of creativity at that level. At that level, they are learning the trade. They are just apprentices. The fundamental idea is that you should be an apprentice before you become a good worker and a master later on. The first thing is to be a good apprentice and do not try to pride yourself on your own creativity. What you should try to do is to learn how to apply properly the tools you are taught.

I was never sure that I would ever arrive at mathematics, invent mathematics, at all when I was in high school or even when I was in the "ecole normale". I was just content in trying to understand what was being taught and to solve the problems given to me. It was only much later when I started to think by myself in trying to solve problems, which have not been solved before, that I began to realise that I could use something to do some new kind of things in mathematics. It is only at that level that you should begin to speak about creativity. Not in the beginning before you reach the highest point (most will never reach it); there are very few people who are creative like Gauss or Riemann. This is unattainable but if you manage to prove in your lifetime one or two nice theorems you should be very happy.

Question. Professor Dieudonné, where would you put future mathematics teachers in your scheme?

Professor Dieudonné. Mathematics teachers are, of course, trained as mathematicians; perhaps not as creative as mathematicians. But they should have a very good background in modern mathematics. As soon as possible. As Professor Bullen emphasized, in places like North America high school teachers are not taught good mathematics at all. They are taught by people who do not know mathematics themselves. Oh yes, I was in American universities for 7 years and I remember the complaint by colleagues in the teacher college that they were taught how to teach mathematics but not what mathematics was. They imagine that if you have a certain number of rules of teaching, you can teach anything and be successful. Complete nonsense.

So I think that teachers themselves should have a very sound training first of all in mathematics. And then they should have additional training, as we do in France. We have a bifurcation here among mathematicians. There is, as Schwartz mentioned earlier, training for the Master's degree in Mathematics in which they are directed towards the ultimate in what they know at that time in order to go into research. And then there is another kind of training which tends to produce future school teachers. After a sound training in mathematics, they are given special training in subjects which would not be needed for research workers but which will be very useful to future teachers. For instance, they are given more ideas on geometry, linear algebra, multilinear algebra using quadratic forms, the relation of geometry with quardratric forms, a little bit of algebraic geometry, a little bit of number theory. And nowadays, after we have been insisting for a long time before this was done, we are teaching them a little bit of history of mathematics. The trouble is that teachers will have to teach things whose origin they do not even know. Well, how were they developed? People may think that the axioms of vector spaces come out of nowhere and were known to people from the time of Archimedes. Well, nobody told them. So they should be taught a little bit of history of mathematics to show how very painful it was for notions to emerge clearly, even in the minds of very great mathematicians. It took a tremendous amount of time and labour, for instance, for the idea of a group to evolve. It took almost a full hundred years from the beginning of the idea of a group by Cauchy and Galois in the 19th century to the first general definition of a group by Weber in 1895. If a teacher has to teach what a group is, he may very well think that in the time of Euler they knew what a group was. And so he has a completely distorted picture of mathematics if he knew no history of mathematics. So we now try to teach him a little bit of history of mathematics, at least a brief outline, so that he should have some idea of how things come into being.

Professor Schwartz. I would like to say something. When I was in high school I had very good teachers. We had what is called "agregation" (competitive examinations) and the level was very high. The teacher must dominate completely the things they teach. They must know more than they teach, and in order that they do not forget they must learn and learn more than necessary. They have to know what are the use of and abuses in modern mathematics. For instance when they teach what is a group or a vector space and if they do not know the general use of groups and vector spaces, they will teach by memory. They teach it formally and for the children it becomes formal repetition. Simply because the teacher does not know more. A clever teacher who is well trained and who really knows what is mathematics and what is physics will teach more intelligently than one who does not know. So one must be careful about that.

Question. Could we hear the opinion of the panelists on the use of mathematical jargon in the teaching of mathematics.

Professor Dieudonné. In my opinion, there should be as little jargon as possible. Have a little bit of it when necessary. You cannot talk of something without giving it a name. But you should avoid all sort of supplementary names which are completely irrelevant. For instance, I am always against introducing unsatisfactory notions, notations and unnecessary words. You should keep the words to a strict minimum. On the other hand, if you do not have certain words, you are compelled to use very devious and long sentences to express something which could be expressed in one word.

For instance, when you are introducing the idea of compactness in elementary topology, "compact" is a beautiful word. If you did not have that word you will be obliged every time to say, "Look here, I am talking about a space in which when you have a sequence which is bounded you can. . ." In this case, once you have understood what is a compact space, that is a beautiful word and should be taught. But there are many words which are completely unnecessary. For instance, at the very elementary level, the poor children are taught what is commutativity, what is associativity, what is distributivity. For them these are completely obvious. Why should they be given to them? Names are given to them when it is necessary. That can happen very easily in the next stage when they are shown what are linear transformations of 3-space for instance, and that when you have two rotations the product is again a rotation. However, when you do it in the opposite direction you get another rotation. At that time it is perfectly natural for the teacher to say, "This is a good example of what is called by mathematicians 'non-commutativity'. And you will observe then that what you have been doing up till now, working with numbers on the other hand is always commutative". At that stage it is perfectly correct to introduce the name. But before that, to children of 12-15, do not quote to them a lot of nonsense.

Professor Bullen. I would like to add to what Professor Dieudonné has said. I found that the people who introduced unnecessary jargon are the people who write textbooks for schoolchildren. My children have come to me with their textbooks and said "What is this?" I do not have the faintest idea. I have been doing mathematics all my life, I have never come across this nonsense. One I remember is "What is the difference between the number and the name of the number?" For a long time this was a problem. This is not just mathematical jargon. It is worse. It is found in many school textbooks, not here I hope. But in North America there are many of these things introduced by teachers of teachers of mathematics which are forced on the children. The children memorized them for one year and then they forget them for life.

Professor Schwartz. This is the same thing in France. What is the difference between sets in "extension" and sets in "comprehension"? One is a set with given elements, the other is a set defined by a property. So in one you have a set defined in "extension", the other one "comprehension". I do not remember what is what. And then if I ask the schoolteacher, "What about the set composed of 3, 5 and the even numbers?", they say: "That I do not know. It is either extension or comprehension." It is a stupid definition but they learn it.

Professor Dieudonné. They also introduced all sorts of unnecessary notations which no mathematician ever use. But they used them in their textbooks, and the poor children have to memorize them. Of course, they forget them when they leave the class.

Question. I would like to ask Professor Dieudonné how the inspector should be trained.

Professor Dieudonné. That is a very good question. Unfortunately nobody can do anything about it. They are more or less self-chosen, self-perpetuating body of people. Sometimes they are very good teachers. They start teaching in perhaps not the top class and then after a while they are noticed by the other inspectors and they are promoted to the higher class at the top in those preparatory schools. And usually they do very well because they are limited to a good programme which they understand. They do not know much about what is used later on. But at least at that level they are good at training people for the professional examination and many of their students succeed in the examination. And they are promoted to the rank of inspector-general. Apparently nobody has been able to do anything about it.

Professor Bullen. One could do something that would abolish them altogether. Many countries do that.

Professor Dieudonné. We would very much like to do that.

Professor Bullen. May I suggest one other problem which has connection with what Professor Dieudonné has just mentioned. It is a problem in our school system. It is that the most prestigious jobs, the best paying jobs, are not the teaching jobs. And so if a person is a good teacher and is therefore some kind of a leader, he becomes a school principal. He no longer teaches mathematics or whatever. Some of the best teachers are taken away from teaching. Somehow we must counter this. I do not know how.

Question. In the syllabus of modern mathematics, set theory is one of the subjects. I would like to know from the panel members whether it is necessary for set theory to be taught in secondary schools.

Professor Dieudonné. There is a great misunderstanding about set theory. Sets are not all of mathematics. Because if you read Aristotle, you will find that he is talking about sets just as sensibly as anybody does now. He even had the embryos of quantifiers and things like that. He knows very well how to represent an object which is now called a Boolean algebra. Now this is what Boole did. Before Boole, of course, people could argue just as well as Aristotle. Good mathematicians, as you can see from their works, never make mistakes in their logic. However, they did not have the concept of sets or rather they did not use it. Aristotle has it in a vague way and, before Boole, nobody used much of sets. One used rather the comprehension of philosophers. They would rather talk of properties; for instance, Galois and Abel never talked of a field. They say, "We consider all the numbers which can be obtained from a given quantity of numbers by addition, multiplication, division and so on." They did not consider a field as a single object. You see from the history of mathematics that it was very hard for people until around 1850 to think of a collection of objects as a single object. It was very difficult. It was even much more difficult to think of a function other than a collection of values but as a single object.

I think the first one who thought of a set as a single object is Gauss in his theory of quadratic forms but it had no influence at all. And the first one to have some influence and who had thought about it more generally was Boole in 1847. Boole was the first to think of sets and to calculate sets, to represent a set as a single object by a single letter. Nobody had done that before. And that was a real, great progress. That is what people mistake for set theory. They mistake Boolean algebra (elementary Boolean algebra) which is only commonsense, as a part of mathematics. We know very well that later on Cantor did achieve and develop a much more elaborate and sophisticated set theory, which really deserves the name of set theory — the theory which takes sets as objects to be studied in detail and looks at their cardinal numbers and their orderings. This is the real set theory. And of course, the fundamental misunderstanding is that people talk of set theory as that kind of thing which is taught in high school. Well, the only thing which is taught in high school is Boolean algebra which is simply a good way of expressing things in simple terms by simple means.

You know how a notation can be fundamental in mathematics. Most people simply do not realize that until the year 1550, nobody could write this down:

$$a_0x^n + \ldots + a_n = 0,$$

They had to write: a number plus, P-L-U-S or its Latin equivalent, or minus and so forth. They wrote everything in long hand; they had no abbreviations. Now imagine what you would do if you were to write even the simplest problem in algebra without using the signs +, x, ÷. Inequalities only appeared around 1650 — a hundred years later. A good notation, a good abbreviation of a long discourse, a good word like compact, are fantastic in helping mathematicians to correlate things. If we did not have the usual algebraic notations, algebra would still be in its infancy. Nobody before Leibnitz was able to write the simplest equation. Before the year 1600 nobody could write the coefficients of an equation like the above. Even the "x" was difficult to come by. The coefficients were written as letters instead of numbers. Imagine what algebra was at that time. The algebra of Michael Stifel who was a populariser and a very good mathematician around 1550, takes 200 pages to discuss the second degree equation. He had no symbols at all. One of the first to introduce indices or powers was Stevin. This was only possible around the year 1700. What would Galois have done if he did not have these notations? He could not have started his theory of equations because he would not be able to write down the general equation.

So when we teach children signs like these: $\subset$, $\cup$, $\cap$, $\in$, it is useful in reducing all sorts of sentences to short formulas. You know the famous syllogism: Socrates is a man; man is mortal; therefore Socrates is mortal, is simply written as: $A \subset B \subset C \Rightarrow A \subset C$. Well, I think it is better to write it that way than to write down a long syllogism as Aristotle and the others did. You see, the introduction of a few signs like the introduction of good words like compact, connected, or rings, fields in algebra, has helped mathematicians a great deal. This prevents them from taking too long a time to understand a long sentence. It has a single formula, a very short formula. They have been trained to understand them. So I think it is a very good thing for children to learn to manipulate these simple signs, which is just commonsense. Of course, nobody should ever speak of the cardinal of a set, not even an infinite set, to children before they attain the age of 15 or 16.

Question. Where does probability fit in your scheme for high school mathematics?

Professor Dieudonné: Probability should be taught as soon as possible. I am not a specialist in probability, but I believe in probability very much. And anytime I encounter a man who has no mathematical training, talking nonsense about statistics and probability, I say it is a pity that this man did not have a good training in elementary probability. So this should certainly start at the most early age. Tell them about dice throwing, for instance, and show them how things occur, heads and tails and so forth. And then gradually improve and become more sophisticated as the mathematical tools themselves become more sophisticated. You obviously cannot do probability in a good way as you do now until you have calculus as a theory. But you can do a lot of things before that on a much more elementary basis, tailored to the mathematical sophistication of the pupils. But you should do probability as soon as possible.

Question. Are there ways to discover mathematical talent or does it come out by itself?

Professor Dieudonné. You see, a good teacher would immediately notice in his class that he has a boy who is mathematically talented. Even earlier. Of course, there are many famous cases. You know probably all the famous stories about Gauss. He was ten years old, I think, when he was given by his teacher in primary school the following sum: add all the first one hundred whole numbers, because the teacher wanted the children to be quiet so that he could read the newspapers during this time. So he thought the children would take a least one hour to do that. And then after five minutes, the young Gauss turned in his slab on which he wrote a single number, and put it on the table of the teacher, and say, "Here it is." Everyone went on computing and everyone went wrong somewhere, except Gauss. The teacher was a bit surprised and asked "How did you do it?" Gauss said, "Sir, I saw that if I take 100 and 1, I get 101; if I take 2 and 99 I also get 101; if I take 3 and 98, I again get 101. So I get all the time 101. And finally I take the product 50 x 101." He did not know algebra at that time, of course, and he got the right answer. The teacher immediately discovered that he had to deal with a pupil who was quite intelligent.

The same thing occured with Élie Cartan who was the son of a poor blacksmith in the French Alps. And when he was about ten years old, the teacher noticed that he was a very bright boy. When an inspector of primary schools came, he wanted to show how good his student was. He sent Élie Cartan to the blackboard. And Élie Cartan did fireworks. The inspector was so impressed that he managed to have a scholarship for Élie Cartan. Otherwise Élie Cartan would never have the education necessary for becoming a mathematician of some fame.

These are exceptional cases. So it is very seldom that a boy, even a very talented one, is noticed as a future mathematician. In France usually around 14. In my case, I really became interested in mathematics at 14. Chevally was a bit earlier. I think he read calculus when he was 12. I am not sure he understood everything very much. I read calculus when I was 14 or 15. I did not understand very much. Anyway, really talented mathematicians can be discovered very easily, provided, of course, they get the opportunity.

Professor Schwartz. I think it depends on the country. In a country where the majority of the people are peasants, a mathematically talented person in an obscure village will never be discovered. But in a country in which there is a strong system of schools, in which most of the people are educated and in which primary school teachers know something, real talent in mathematics is visible. People see it anyway. But it depends on the general education of the population. If it is there, I think good talent will be seen.

Professor Dieudonné. Provided the boys have access to the knowledge. I am quite sure that there are countries in which there are potentially very good mathematicians who will never be discovered because the economic situation and the school situation are too bad. And they will never be able to have access to the knowledge which you have. Of course, there are stories of people, of boys who invented mathematics. This is completely false. Even Gauss did not invent mathematics. The story in France about Pascal, which says that Pascal invented geometry by himself, is completely false. It is a legend. What really happened was that he was given a book of Euclid and he read it all by himself, which is already something if you are a boy of 12. It is certainly not true that he invented geometry by himself.

Professor Sen. It is already time. It is my last duty to thank the members of the panel for the extraordinary talks that they have given today and the discussion it generated.