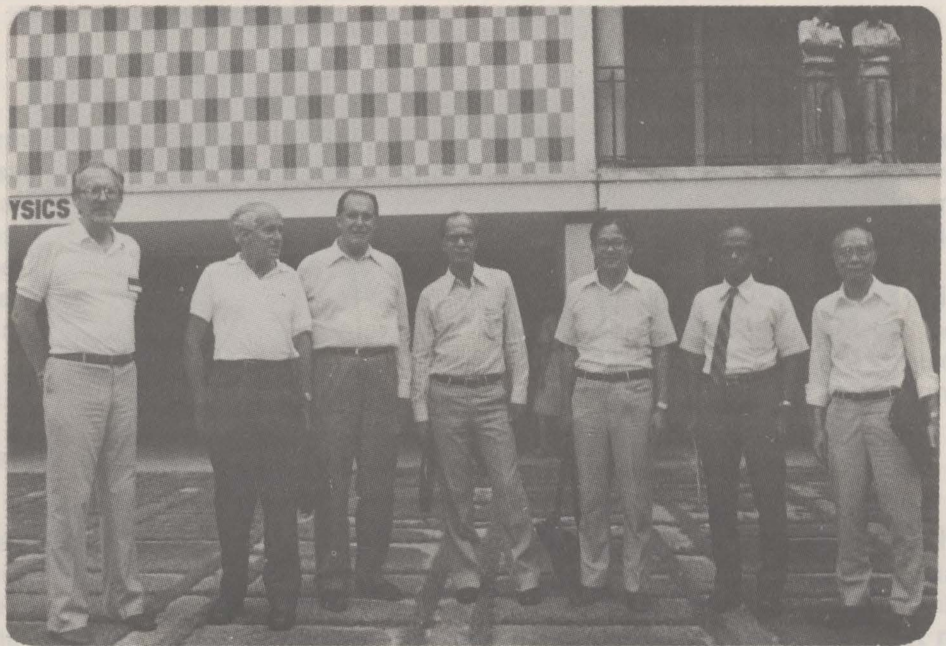
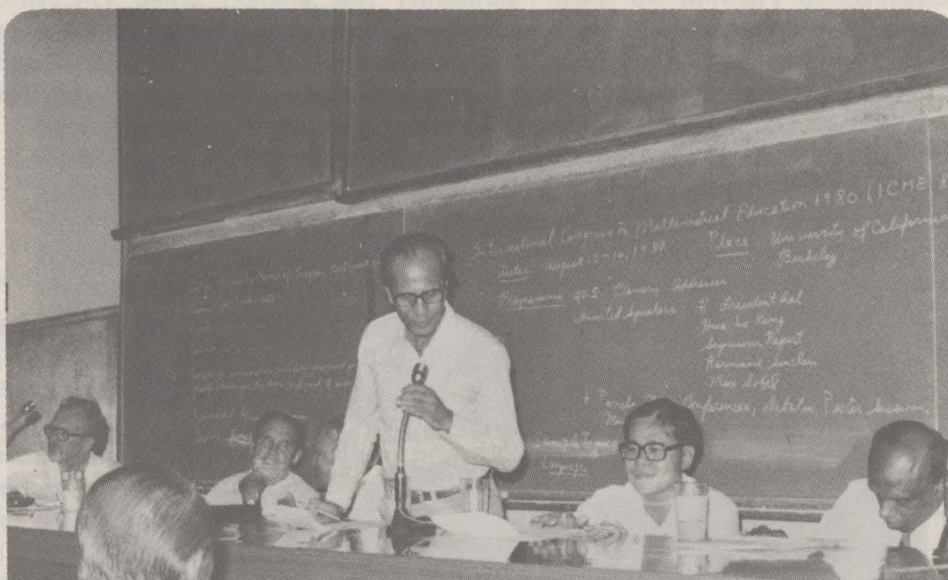


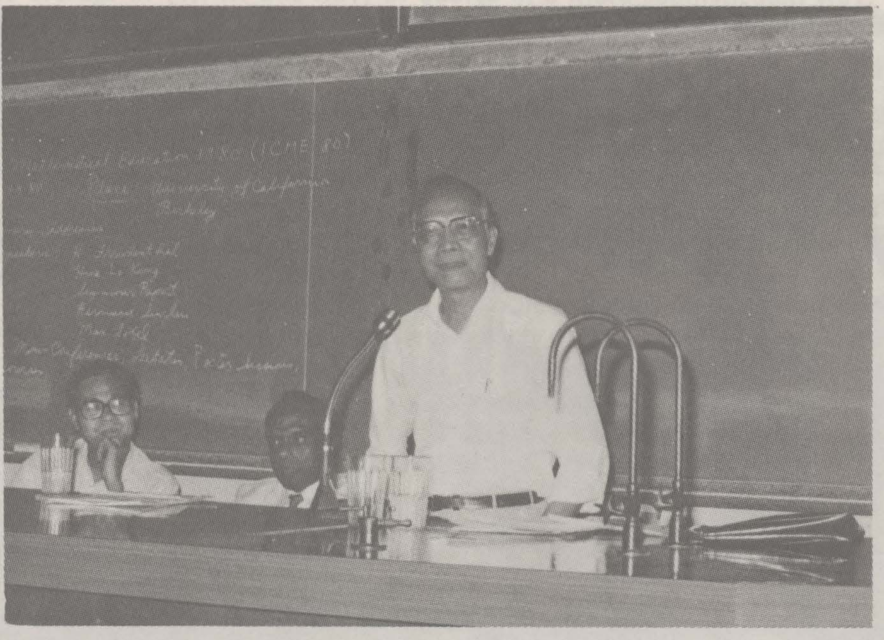
(Courtesy *Y.K. Leong*)

The Panel: (left to right) P. Bullen, L. Schwartz, J. Dieudonné, K.K. Sen (Chairman), W.S. Tan, S. Thiagarajah, Y.C. Wong.



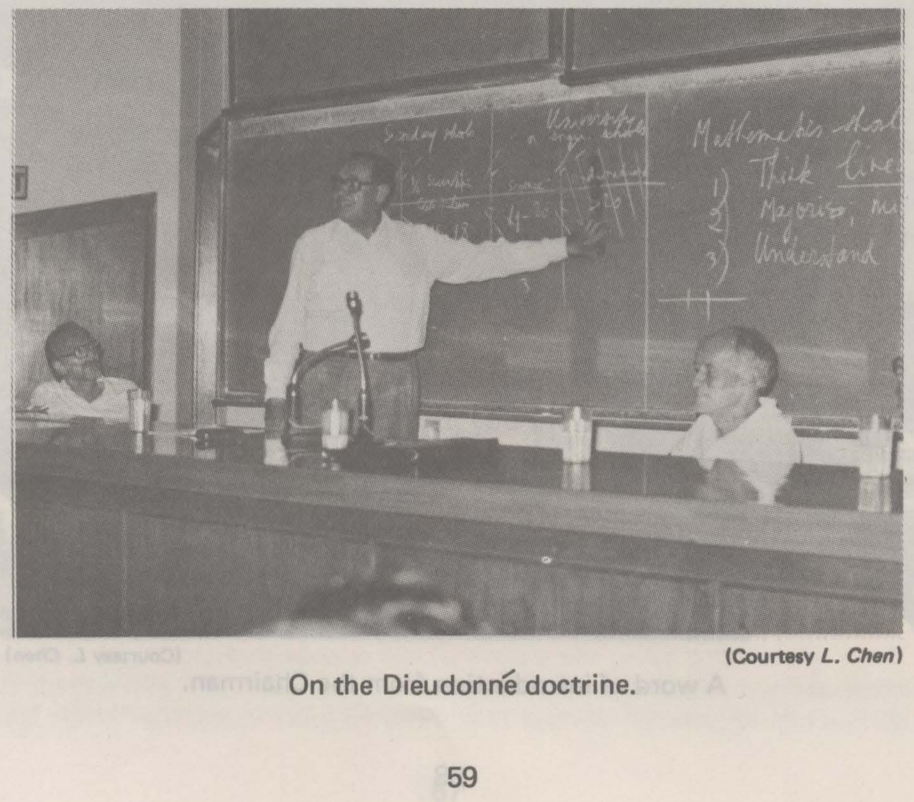
(Courtesy *L. Chen*)

A word of introduction from the Chairman.



(Courtesy *L. Chen*)

On the Hong Kong scene.



(Courtesy *L. Chen*)

On the Dieudonné doctrine.

FORUM ON MATHEMATICAL EDUCATION

Part I

Chairman (Professor Sen). The problem of mathematical education is as old as civilisation itself. This has been discussed in depth in Plato's Academy 2500 years ago, and in the 20th century we are only trying to discover the solutions with more intensity. This discussion in mathematics has to some extent been institutionalised 80 years ago by the setting up of an international commission in 1899 and by the affiliation of ICMI (International Commission of Mathematical Instruction) to IMU (International Mathematical Union) in 1960, and periodical meetings are being held. I understand that our views from Southeast Asia should be represented at ICMI meetings. To the foreign delegates at least, we can convey some idea concerning the problems of Southeast Asian mathematical education. And we would like to learn from them how we should go about solving these problems.

To start the discussion, I will tell you about the procedure. The first half will be devoted to opening statements by members of the Panel. As you can see, we have a very distinguished Panel: Professor Bullen from British Columbia, Canada; Professor Dieudonné, the doyen of French science and one of the founder-members of the Bourbaki group; Professor Schwartz, a Fields Medallist, who does not need any more introduction. All of them have been educators for a long long time. Our Southeast Asian team is represented by Professor Tan Wang Seng, President of the Southeast Asian Mathematical Society and Head of the Department of Mathematics in University Sains, Penang; Mr. Thiagarajah from the Ministry of Education of Singapore, who is in charge of mathematics education; Professor Y. C. Wong, doyen of Southeast Asian mathematicians, who besides being a great mathematician himself, has produced great mathematicians.

I should like to start by asking our Southeast Asian representatives to speak first so that our foreign representatives may know better what our problems are. Our first speaker will be Professor Tan Wan Seng. He will be followed by Mr. Thiagarajah, Professor Y. C. Wong, Professor Bullen, Professor Schwartz and Professor Dieudonne. So I call upon Professor Tan Wan Seng to give his views.

Professor Tan. Thank you Mr Chairman, Professor Sen. Now being the first speaker in a forum is both difficult and easy. It is easy because I do not have to find a solution to them. Now the primary concern in mathematical education is probably the curriculum: what we should teach and how we should teach. In my view, designers of a mathematical curriculum have, in a way, an easier task than that of designers of a history curriculum say. A subject like history tends to mould the character of a person or nation, but the damage of a wrong scientific curriculum is, I would say, not as disastrous as that in the teaching of history. Still, we have a responsibility in designing what we should teach.

I believe that a mathematical curriculum must change with time. It must be evolving all the time to match the progress of the country. But there are many questions to be raised. What are the factors that must be taken into account when designing a curriculum? For instance, it is not necessary to have a very sophisticated curriculum in mathematics in a country which is rural in character and where the principal economic activity is nothing more than bartering. However, in a country that is highly industrialised, a very sophisticated curriculum would be requried to match the needs of the country.

The next question now arises. Having adopted a curriculum that may meet the requirements of a particular country for a certain period of time, how should we change the curriculum to meet the changing needs ten or twenty years later? I personally feel that things should move slowly because if we make too drastic a change, I think it will be quite disastrous. Changing a curriculum is not a simple matter. It involves many parties, first the students whose background you must take into account, their interaction with the environment and their experiences. You may have a very good curriculum but if you lack the experience, it is no use. Because it is not going to affect him, it is not going to make any impact. So I personally feel that if there is a change, it should be evolutionary, not revolutionary. Step by step. Perhaps groups of topics. Put them together, test them out. And in that case you will find that preparing people to teach them would be a easier process. Take, for example, the recent introduction of modern mathematics in the drastic change in curriculum. You have a lot of problems trying to retrain the teachers. Teachers' attitudes towards their jobs and towards mathematics do get fixed after a while. You may have to go slow; I mean a stage by stage development of the curriculum rather than a complete change overnight.

The next question you should ask is whether we should adopt the well-tested curriculum of well-developed countries in the hope that eventually we will match up with them. I think we have to be very careful about that. Their social conditions are different and, in fact, in America they do find that social factors affect performance, even in mathematics.

The other point we would like to discuss is teacher training. Having adopted a curriculum and having decided on the changes to be made, how do we go about training teachers? Now mathematics advances very fast. As Professor Lions mentioned, even in numerical analysis things have changed dramatically within 20 – 40 years. The curriculum in France is highly sophisticated but we are more familiar with the British type of mathematical education. How should teachers be trained in order to be brought up-to-date? I think it is necessary that teachers be required, not as punishment but inducement, to update their knowledge of the type of mathematics that are experimented in other countries, and perhaps their points of view would be very useful. We cannot adopt a curriculum in a vacuum. You may propose something, but to implement it is another thing. And teachers are the ones who are implementing it. If you can have some sort of continuous training whereby teachers go back once in a while to update their knowledge in a slow process of preparation for new changes, it may be more effective than having a sudden change in curriculum and then putting all your resources together to try and train the teachers to meet the needs of the new syllabus.

The other problem that we can talk about is public examination. Public examinations tend to have an effect on teachers. Examiners play tricks; hence teachers teach tricks and less mathematics to help students overcome the tricks. After a while, students do better and examiners play more tricks, and teachers learn a bigger bag of tricks. And so it goes on. Now this is the type of problem which we may have to look into because if I am not wrong, the original idea of introducing modern mathematics was that you should teach more mathematics and a more logical approach in mathematics, rather than bags of tricks. However, after a while, teachers get smart and they tend to teach tricks even in a modern mathematics curriculum. I think it is quite plain that teachers are now very familiar with things like sets, transformation geometry, number systems, and so on.

The next thing I would like to talk about is the level at which we should teach certain topics. This is a difficult question because at the moment we are not quite sure how children develop. I have seen topics like number systems being taught to children at the age of 14 or 15. They are expected to be able to set up a one-to-one correspondence between the real number system and the number line and to see that it is not possible to have a one-to-one correspondence between the rational numbers and the number line. When I was teaching a Form II class, I once said it was not possible to set up such a one-to-one correspondence. They say, "How can there be gaps? You told us a few days ago that between two rational numbers there is another rational number." Now this is the type of things which a child of 14 or 15 is unable to cope up with. So we have to discuss problems like this.

The other point I would like to raise concerns certain classical topics in mathematics which have recently been phased out. I refer, in particular, to geometry, things like Euclidean geometry. I like to think that in Euclidean geometry, students learn about a formal proof, and the construction of a proof. Now we have introduced more transformation geometry in deriving the properties of various geometrical shapes and figures. It may be interesting to debate whether children learn more through transformation geometry than through Euclidean geometry. Probably Professor Wong can answer this question.

Finally, I think that in constructing a curriculum, another point to bear in mind is never to take only the views of mathematicians because mathematicians tend to think more of mathematics than of the applications. In pre-university classes, students are being prepared to join the university but most of them are not going to do mathematics. They are going to be engineers, economists, managers, physicists, chemists, biologists, pharmacists, doctors, and so on. They also need some practical mathematics. So we also need to seek and discuss the views of other people: what they actually need rather than what the mathematicians want. I will now leave the discussion open.

Mr. Thiagarajah. Mr. Chairman, ladies and gentlemen, I think it is indeed a great honour to be chosen to sit in a panel composed of half a dozen eminent mathematicians. At the same time I feel somewhat diffident and nervous when I compare my rather limited mathematical experience with their great mathematical achievements. But the thought that I am closer to the teachers and students in our schools than the other distinguished members of the panel gives me courage. I will make sure that I confine myself to the only area I am more familiar with, namely mathematical education in Singapore schools particularly secondary schools.

Until 1972 most Singapore schools followed a traditional syllabus in mathematics patterned on the Cambridge 'O' level Syllabus B in mathematics. In 1973 a new syllabus was introduced. The new syllabus for Secondary 1 and 2 contained some modern topics and was common to all schools. As for Secondary 3 and 4, schools were given the option to choose between two syllabuses, the first was fairly traditional and resembled the Cambridge Syllabus B in mathematics, the second was more modern and resembled the Cambridge Syllabus C in mathematics. Schools were completely free to choose either syllabus.

According to the Ministry of Education time schedule, the new modern syllabus which was introduced in Sec 1 in 1973 was supposed to be taught in Sec 4 only in 1976. However, it speaks well of our teachers to note that some schools switched over to Syllabus C as early as 1971, and by 1976 nearly 50% of the pupils offered Syllabus C at 'O' level and in 1978 nearly two-third of the GCE 'O' level candidates offered Syllabus C.

Since 1976 our local universities and other tertiary institutions have been getting pupils who have gone through the C Syllabus in increasing numbers. The mathematics departments of these institutions should be able by now to compare the performance of pupils who have gone through the new Syllabus C and those who have gone through the earlier Syllabus B, and tell us who are mathematically better prepared for advanced mathematical studies at higher levels. If such a comparative study has not already been done, it must be done without further delay.

I tried to compare the examination results of candidates doing the C Syllabus and those doing the B Syllabus to see whether that will shed any light on the subject. Over the 8-year period (1971 to 1978) the average pass rate for candidates offering Syllabus C is 58% while the average pass rate for those offering Syllabus B has only been 45% – a very significant difference. From this one should not jump to the naive conclusion that Syllabus C is better suited to our pupils than Syllabus B. I think the difference in pass rates between Syllabus C and Syllabus B candidates is more due to the type of candidates than to any other factor. Candidates offering Syllabus B are generally the weaker pupils. Many teachers seem to feel the Syllabus B is more suited to weaker pupils than Syllabus C. However the results do not seem to bear this out.

If we look at the combined pass rates for both Syllabus B and Syllabus C candidates, we notice that it has remained consistently around 50% for the last 10 years showing that the change in syllabus has not affected the overall performance significantly. But the fact that only about 50% of the students pass GCE 'O' level mathematics in either Syllabus B or C is a cause for concern. If somebody fails in a subject after studying it for 10 years, we can only draw the following conclusions.

(a) The pupil has no aptitude for the subject.
(b) The syllabus is beyond the reach of the pupil.
(c) The teaching of the subject is not effective.
(d) A combination of the above factors has caused the failure.

I personally think that a combination of the above factors are at work. Some of the pupils who have failed in mathematics are not really attracted to mathematics at least in its present form: both content and presentation. They are offering the subject because it has been made a compulsory subject for 'O' level since 1975. If mathematics is not compulsory these pupils will drop the subject with great relief to themselves and their teachers.

Others will continue to do mathematics even if it is not compulsory, but they will not make the grade unless the syllabus is tailored to their needs. It is also a sad fact that a number of students fail simply because of poor teaching. Both the content, knowledge and methodology of many teachers need improvement. To be an effective teacher one must have a clear understanding of what one is trying to teach and must have the ability to put it across to his pupils in an interesting and lucid manner. You can make the simplest idea sound mysterious and baffling if you are muddle-headed yourselves and your presentation is sloppy and slip-shod.

Some of the failures are due to our system of class teaching which tends to become highly impersonal in our invariably large classes. As you all know mathematics is a highly structured subject with a hierarchy of concepts and pupils must understand and master each step before the proceeding to the next. Unfortunately in our system where pupils are promoted to the next grade on the basis of overall performance and not on performance in mathematics, it is possible for pupils to go to the next higher grade without successfully mastering the mathematics of the lower grade, resulting in a big gap in the pupils' mathematical knowledge which grows wider as he goes higher. How many of our teachers ever take note of this and do anything about it? Perhaps it is not fair to blame the teachers. It is the sytem — the class teaching system where the lessons are pitched to an imaginary average pupil who is supposed to have reached a certain pre-determined level of mathematical proficiency which is to be blamed. It is high time some of our schools try out the concept of banding or setting for mathematics where pupils study mathematics at their own pace irrespective of the classes they are in. Programmed learning is another useful idea which few Singapore schools have tried out.

As for what mathematics to teach, I do not want to enter into a debate here. Personally I think that the question of how to teach mathematics is even more vital than what mathematics to teach. Any mathematics, ancient or modern, which will make pupils think creatively, any mathematics that will help pupils to appreciate the beauty and power of mathematics, any mathematics that will inspire pupils to formulate new problems and solve them originally is good mathematics.

Professor Wong. Mr Chairman, ladies and gentlemen. The problem of mathematical education in Hong Kong is very similar to those in many other countries. However, as a place, Hong Kong is quite unlike many other countries. It has only about 400 square miles of land with nearly five million people. We do not have enough schools or university places for all those who want to study and therefore competition for entrance into schools and university is very very strong. The selection of students in university and schools is based mainly on examination results. So there is a great deal of emphasis on examination tricks. This is not a good thing at all. We have different kinds of examinations. For the English schools, you have 5 years of high school and then 2 more years, after which you take the advanced level examination.

On the other hand, in the Chinese schools, you have three years of Junior Middle School and three years of Senior Middle School, and at the end of it, you take an examination.

There is an examination authority which runs all the examinations in Hong Kong. For the selection to the university, Hong Kong University has advanced level examination, and the Chinese University has their own entrance examination. So the syllabus for the examinations is very important. The people who are responsible for setting the syllabus for the mathematics examination would have to be very careful in setting the right kind of questions. Usually a committee consisting of university lecturers, school-teachers and maybe a few others, is responsible for setting the syllabus, which has to be approved by the authority. Then they set the question papers for the examination.

Now because Hong Kong is administered by the British Government, the educational system follows more or less the British system. In the early days textbooks are actually those written in England for English students. Very often we use these textbooks and sometimes we translate them into Chinese and teach from them. As a result students in schools have to spend a lot of time, for example, in converting British currency to Hong Kong currency. And some exercises are set, of course, on the playing of cards, and very few Chinese students play cards. This situation has, of course, gradually changed. Nowadays, we have textbooks written by Hong Kong people specially for Hong Kong students. That is a great improvement.

The second problem, of course, is the problem of modern mathematics. About 1960 the University began adding to the syllabus just a little bit of sets and later on a little of vectors and matrices – just a little bit of linear algebra. At that time, we felt and still feel that you cannot change the whole syllabus even though it would be a good thing to do so. We contend that much of traditional mathematics should still be there. Some of the topics may be replaced by more useful topics, but certainly the main body of the syllabus is traditional mathematics. There are, of course, people who become too enthusiastic about this kind of new mathematics, and some textbooks were written before much preparation was done. As a result you get much garbage.

One problem we are facing now arises from the trend to have a unified course in mathematics, not the compartment type of mathematics – arithmetic, algebra, geometry and so on. We want a unified course, combining the basic mathematics, arithmetic, algebra, geometry and so on. But I think we do not yet have a very good kind of arrangement that will make the course really interesting, easy to understand and easy to teach.

Another problem we are facing is how to cope with students with different ability in mathematics. Some may be very good, some very poor. If you teach them the same thing at the same place, the result is unsatisfactory. Should you offer different courses to students with different interests? For example, some students want to study arts subjects and some want to study engineering or science subjects. Are you going to give them the same course in mathematics? These are problems common to all countries.

Now we come to the university level. I have been in Hong Kong University for 28 years, so I can see the gradual changes. My own view is that the goal of mathematics education is that after the students have completed their mathematics courses in the university, they should be able to pick up whatever further mathematics they need in their particular kind of work. They should also be able to solve problems, as they arise, whether mathematical or not. Hard work and talent are important, but knowledge and wisdom are not synonymous. How to achieve these goals? It is not easy because we cannot possibly teach our students all the mathematics they need to know. A selection must be made. So what kind of mathematics to teach our students and how to teach it? I think it is a big problem. In Hong Kong we have the so-called lectures where the teachers teach students either by using textbooks or by lecture notes or by some other means. And then we have the exercise classes. In other words, you have three hours of lectures and one hour of exercise class. When the students are assigned problems to do, they may be required to do them on the blackboard. For the higher classes we have the so-called seminars. Each student is assigned a very easy mathematics paper from the American Mathematical Monthly, for example, or things like that, and he studies his paper very carefully and report it to the class. And the students together with the staff observe his presentation. In this way, a student learns how to go about finding the material.

So this is the way we are doing things and I think the system can still be improved upon. We would like to hear what the opinions of the other speakers are. Thank you.

Professor Bullen. Professor Sen, if I understood Professor Sen's introduction, he was hoping that I, as a representative from North America, knowing all the problems, will be able to help solve them. I find that the problems are the same all over the world and the only difference is that we have not solved them longer than you have not solved them. Sometimes conferences on education such as this one get people together who have solved problems locally and whose solutions may be transferable. I will give a simple example of this.

At the last ICMI Conference at Frankfurt, perhaps one of the most exciting things I have gained was an interjection from the audience by a professor of education from Australia, a woman who has been studying the difficulty of teaching mathematics to aboriginal students. Now she came out with a theory as to why these students could not do well in mathematics, which she thought could be applied to a wider context. Now I find the theory applicable to normal drop-out students in the North American context. How valid that was I do not know. But it was a possibility and it could be helpful elsewhere. So I again say that the experiences of solving problems which seem to be common must be shared. In this way we may be able to help each other.

I find that there are two groups of students. We sometimes talk about both of them together and this causes some confusion. Now students who will be our future leaders of society, the business executives, professors, lawyers, doctors, are very strongly self-motivated students. And then there are the majority of our students who will be citizens. The needs of these two groups are entirely different, and much debate is caused by our confusing them at various stages. For the first group of students who are highly mtoivated and who know where they wish to go, I think there is not so much difficulty about the school system. We can certainly discuss in great detail as to what exactly the syllabus should be, but most societies, I think, seem to be very successful. As far as I can see this morning, higher education in France is very successful in training mathematicians. And it is true in the Soviet Union and the United States where more and more excellent mathematicians are being produced from very different educational systems.

However, I think we are not very successful in dealing with the second group. A large number of future citizens, I think, need a mathematical background to be able to cope with the society in which they live. And I think in what I will call the overdeveloped countries, the century of universal education is a complete failure. I am not convinced that there is much evidence that we, as citizens who have twelve years of compulsory education, are more able in coping with the society in which we are living than the uneducated (in a formal sense) citizens of a hundred years ago. And I think this is particularly true in mathematics. I am not sure that the majority of the people whom we turn out in our school system can cope with the mathematical demands which society puts on them and which are minimal. In particular, to know when your trade union leader and your politicians are using statistics to bamboozle you, to basically lead you on. This is an essential tool of a modern citizen. He must be able to cope with elementary statistics and to know what is meant by doing 600% better than last year and therefore we can claim a 600% wage increase. We all know that is nonsense but nonetheless that kind of argument is used all the time and is taking in members of our society. Price variation, percentages are all every elementary ideas. Coping with income tax forms: I do not know what it is like in Singapore, but I marvel at people who have no higher education and who can cope with them. And equally important is the role that computers play in our society; the ability to understand that role and to cope with them. These are by mathematical standards very elementary and almost trivial concepts. We in North America are not succeeding in training citizens who are capable of facing the stresses of modern society. I do not have any solution for that problem.

I do think that university professors have a very very serious role to play. We hear professors saying that the high schools are not doing a good job. "Look at the people who are coming in. They know less than they did a few years ago." I am not going to dispute this. But I know that in our university, in the third year, we have professors who are complaining that students still do not know anything. But we have had them for two years. Now whom do we blame? So this is the process of what we call "Passing the buck". Let the next stage fail them. Not this stage. This is particularly serious, I feel, in the training of teachers, which is exceedingly neglected, probably very much neglected in North American universities where somehow or other it is slightly "infradig" but slightly not below your notice to train teachers. Teaching training is something done by bachelors of education, real academics pay no attention to this, and this is a crime as far as I am concerned. Then we wonder why our mathematics teachers know no mathematics. I am sorry, I do not wish to insult the audience. I am talking about mathematics teachers in North America. We train teachers in a very bad way. We should not be teaching tricks but thought processes which is what modern mathematics is about. We certainly go to the undertrained people, overworked as well by many standards, and suddenly say, "Ha, ha, ha, tomorrow there will be a new syllabus. All we have learned up till now is unimportant." And then you wonder why modern mathematics fails.

So we have these problems. Let us try and solve them. I would like your help and I will certainly do anything I can to help you. Thank you.

Professor Schwartz. Well, I do not want to say much because I have already spoken this morning. I just want to stress three points. I completely agree with what Professor Bullen said.

It has been traditional in France that the secondary school teachers have been completely trained by the university professors. The secondary school teachers in mathematics are trained by the university professors in mathematics, and we will never abandon these privileges. There were attempts by the ministers to take away these privileges from us but we resisted. Instead of having one minister for education, we have now two ministers: one for higher education and the other for primary and secondary education. There is probably an explanation for the high level of our secondary school teaching in the past. It has been one of the highest in the world. It is the same in Germany but much higher. Secondary school teaching in the United States doe not depend on the university professors. Our secondary school teachers have to prepare for the "agregation" which is a competitive examination. Now competitive examination is more useful for doing research than for training teachers in secondary schools. And we have a lower grade which is called "SST" which means it is at a lower level. And it is sure that teachers in our high schools are presently at a lower level than they were when I was in high school. That is surely an explanation for some failure in secondary school teaching. And so I supplicate you, in all countries, to maintain the direction for the training of teachers in high schools by university professors. And it is absolutely essential.

The second point is the following which Professor Bullen has mentioned. To cope with this world we live in, it is necessary to know science. A pure education in humanities is not sufficient. It is good to have humanities. It is necessary to know humanities: literature, arts, history, geography and so on. They are indispensable. But it is impossible to cope with the world, to have a good situation in the world and to have an interesting life, if you completely ignore science. We have to do something about that. After all, we resign too easily to the fact that too many people ignore science and too many people prefer to ignore science. They are proud of that. Many people, when they learn I am a mathematician, say, "Oh, I am very wea in mathematics." I always tell them, "You are not to be proud of that." Wha would you say if I say that I cannot distinguish two elementary things in arts oı literature or economics or social sciences? You have to know something in mathematics. Maybe you are weak but you must regret it. You must not be proud of it. You must train your students to know that they have to know something in science. Let us take an example which may be familiar to all of you: to be able to read and write in Chinese. After all, it is very difficult. However, people who live in China have to do that. They cannot go into the world, they cannot succeed, they cannot have a culture if they ignore the need to read and write Chinese. So it is compulsory in some ways. One trains them, educates them in the fact that they cannot avoid that. In some way, we should do the same for science, and, in particular, for mathematics. Tell people you cannot live in this world if you ignore everything in science and make that understood by people. As a consequence of the first point raised, we have not succeeded.

The third point is that we must motivate people in science, mathematics in particular. To motivate is not just to teach something which is memorised, and there are now in our teaching a lot of things in mathematics which are pure repetition. For instance, they have to repeat the, I do not know how many, axioms of a vector space. That is nothing to them. We must show them that mathematics is useful. If we tell them that it is indispensable then you must teach mathematics. Besides, it is interesting also. They know that it is not indispensable in life to know what is a vector space, but they must know other things. To know some geometry, to know some algebra, to know some elementary rules of analysis. And, for instance, to compute income tax, they know there are a lot of mathematics which are indispensable and more practical. Well, these are the three points I want to stress.

Professor Dieudonné. I will be more ambitious. We have a big problem which, as Professor Sen said at the beginning, goes back to Plato and perhaps earlier of what should be a mathematics education and for whom. Well, let us try to analyse this problem and to study it in a logical way.

Mathematics as a tool or cultural ornament? We have to face that question. Well, I, of course, think that it is false and I think that every mathematician thinks that. It is one of the jewels of our present culture for over 2500 years and it is still evolving, as I told you this morning. On the other hand, it is quite clear that at various levels it is a tool for many people. Sometimes I read an article by Thom where he emphasized the role of play of games in mathematics, that mathematics should not merely be a tool for a given purpose. Allow some idea of play of games. Well, it is not wrong in that sense but if you take that point of view, if you emphasize mathematics as a cultural ornament, you are immediately open to attack by a lot of people who will say that they also have cultural ornaments which are certainly as valuable as mathematics. And certainly I will be hard put to answer a man who tells me that Greek poetry or Chinese painting is just as good a cultural ornament as mathematics. What can you do after that? I cannot. So if we rest our case on the value of mathematics as a culutral ornament it will certainly be weak. So I will deliberately leave that part aside although it has its importance. But for mathematics education, I think the emphasis should certainly be on the idea of a tool. Now, tools to whom and for whom and for what?

Let us try to follow what children study, starting at the age of 12 and ending beyond 20 and what they will do in later life. I think one can divide it into four stages: half in secondary schools and half in the university.

Secondary Schools		University or Engineering Schools	
Common curriculum	1/2 scientific curriculum	Science	Mathematicians
$\sim 12-15$	$15-18$	$19-20$	>20

The first one is approximately from 12 to 15. I think everybody agrees that the mental capacity and mental development of the child is not enough for specialisation at this time, otherwise it will result in probably a lot of errors. So it is commonly thought, and I think it is correct, that this should be a common curriculum for everybody. Around the age of 15, the boys and girls are already more determined and they have natural tendencies. Some of them will be interested in science. It is quite natural. I will not agree with Schwartz that everybody should be interested in science. After all, there have been excellent minds in the world, philosophers and poets, who have never had any interest in science. I do not see why these people should not have the right to choose culture. So I am not very strongly inclined to impose science on everybody and it is at this age that there will already be a divergence. A large part will not choose science. And I think they have the right to do so. About these, of course, I will have nothing to say. They are not interested in science; why should they have mathematics?

I now concentrate on those who have an interest in science. This is the next group who are still in secondary school, aged 15 to 18, and who have a definite interest in science and therefore have a curriculum half of which is scientific. It will be a pity if children up to 18 are not confronted with all sorts of other pursuits and other topics which are certainly part of our daily culture. But certainly the emphasis for these people will be on science. At 18 then, the entrance to university or engineering schools. Well, at that level again there is a lot of divergence. Even those who are interested in science will not necessarily follow a career which will need a lot of science, especially mathematics. So a lot of people at that level will go into business schools, commercial endeavours and so on, which have not much to do with science anymore. These are the people whom I will neglect completely. For those who enter the university and definitely want to learn a very serious curriculum in science, this is roughly what happens in the first few years of university, say 19 or 20. And then after these two years, there is again a lot of divergence. Those who go into science take mathematics, physics, chemistry, biology, and so forth. These I again reglect. I arrive at the mathematicians, by which I mean those who will make mathematics the central theme of their life, whether they be teachers in the secondary schools or in the university. I think once we have that, we can argue in a much better way because, if we admit that mathematics is a tool, then we have to start from the right and go backwards.

If we teach something here, what will they need to understand what is being taught and so forth. And so we go in a regressive way and try to see what kind of necessary topics should be taught and what type of teaching should be given. In other words, I am emphasising here that mathematics education should be taken as a whole, and the fundamental idea should be continuity of teaching. And I have observed that in many curricula plenty of things are taught which will not be applied later on any more. Nobody knows why. This I think is completely wrong. The main idea should be continuity of teaching. Whenever you teach something you should say what the use of that will be later on.

So the first thing I will do is to disregard the last group. Teaching of mathematics is something in which there is practically no problem because at that stage people specialise in mathematics and it is wellknown what should be taught to them

The mathematics is there, it has to be learned in some way; people will have to learn to do problems. So this, I suppose, is agreed upon by all people. The difficulty starts when you go backwards. I think one of the first principles to be derived from this scheme is the one I am going to write down, and this applies to the first three stages only: "Mathematics should not be taught to future mathematicians."

After you have gone through the first three stages, what is the percentage of population left here? Maybe one in a thousand or one out of 10,000. So should you gear your whole education at all the expense and trouble for 1000th or 10,000th of the population? Schwartz argues very persuasively for this – that there should be a very good teaching for future mathematicians. Alright, I do not disagree. But put them aside, put them in special schools. No need to talk about mathematics education for these people. The problem will resolve itself. When there is a real future mathematician, and that is usually discovered around the age of 15 or 16, even if you show him anything in the worst possible way he will seek his way through. There is no problem about that. The problem is for the 999 out of 1000. What should be done for this group of people? That is the first principle: "Mathematics should not be taught for future matematicians." Let us see what the consequences of this are. No projective geometry, non-Euclidean geometry; no definition of $\mathbb{R}$ by cuts; very little abstract algebra, abstract topology. These are all for the last stage. And even more so for others. This excludes the fundamental principles and all sorts of nonsense taught in many universities. Why should abstract axiomatic projective geometry be taught to people in the first two years of university? This is perfectly useless even for mathematicians.

On the positive side teaching in these three stages can be summarised in three principles. Through all stages, these principles are valid from beginning to end. (1) Think *linearly.* (2) Majorise, minorise, approximate. (3) (Here I agree with Professor Bullen.) Understand probability.

Let me elaborate a little bit on this. "Think linearly": What does that mean? It is a fact, maybe a defect of our brain, that at present at least 90 percent of mathematics is linear. We do not understand non-commutative mathematics. Whenever we are confronted with a problem in non-commutative mathematics, most of the time we try to linearise it some way or other. This is fundmental, and in all the topics which I mentioned this morning this can easily be seen. We are in an era of linear mathematics. I hope that 200 years from now our descendants will be more clever than we are, and be able to tackle non-communtative mathematics on its own merit better than we can now. On the other hand, the fundamental idea of linearity pervades all of modern mathematics. I could give you a lot of examples. That will take me too far. Let us take only elementary examples. All the equations of mathematical physics which we can tackle are linear, not to speak of matrices which are used everywhere.

"Majorise, minorise, approximate." This means that we should teach as soon as possible to all students that there is no such thing as a real number. What exists is an interval where some real number will be, and that is the only thing you can give. We practically never deal with the real number. We deal with the real number up to an approximation. And so the idea of an approximation, the idea of error, should be the goal of the whole structure of mathematics education. Children should be taught as soon as possible to understand that you never get the number except within a certain error. And when you work with numbers, when you compute with them in any way, you have to evaluate what happens to the errors. That is why you have to majorise numbers, to minorise them and to approximate them and to know every time what kind of approximation you are getting. Within what limits is what you are doing true? So this is the second fundamental idea.

The third has already been emphasized by Professor Bullen. There is no need for me to go beyond that. It is perfectly clear. It is a fundamental necessity.

Let us work out in more detail the first three stages starting from the third and working backwards as before. What should be done in Stage 3, the first two years of university? Well, from the linear point of view we should have, of course, linear and bilinear algebra, over the real or complex numbers. On the other hand, n-dimensional is quite essential. So linear and bilinear algebra should be given in as much a geometrical language as possible. This is again something which one must insist on all the time. As I have said the other day, there is no such thing in mathematics as algebra, geometry, etc. Everything is unified. At that level and other levels as well, you should never deal with algebra without geometric interpretations. You should never deal with geometry without algebra. In other words, there should be complete fusion of linear algebra with geometry. I have written a book several years ago which tries to do that by showing that every algebraic technique of elementary linear algebra has a geometric counterpart. Sometimes several interpretations. For instance, a system of 3 linear equations in 3 variables: what does that mean? If you keep it that way, it is stupid. If you interpret it geometrically, it means several things; for instance, intersection of three planes. Or you can write it in a natural way and then the problem is: given a vector in any subspace and a basis of this subspace, find the coordinates of the vector relative to this basis. It gives rise to a linear space, find the image of this mapping. All of these boil down to the same problem of solving a linear system of equations, but we have three interpretations which are geometric. I claim that everything in linear algebra can be interpreted in at least one geometrical way. So teaching algebra without geometry is nonsense, teaching geometry without algebra is also sheer nonsense. That is the trouble with the old-fashioned geometry of Euclid, which fortunately has disappeared. Fortunately because it is the worst possible technique you can get. When you want to prove a theorem as simple as Pythagoras' theorem, you have to divide the thing into, I do not know how many, triangles for no special reason, and of course, the normal reaction of any sensible boy would be that it is a bag of tricks. Whereas once you have at that level the notion of a vector space and bilinear forms, Pythagoras' theorem is a triviality. We do it in one line and so it should be.

And, of course, in the development of this idea in linear algebra, you will need all sorts of algebraic structures such as groups, rings, the ring of endomorphisms of a vector space. They are very beautiful geometric objects on which you can give a lot of interesting problems. You will meet the group of invertible elements of a ring. The group of automorphisms of a vector space, the general linear group GL(n), is one of the most important objects in modern mathematics and is fundamental for everybody. You will need its subgroups, the orthogonal group, which is at the basis of Euclidean geometry, and so on. Any time you need that, you can, of course, introduce the concept of groups. In that particular case, there is no need to go into the complete study of abstract linear groups. It is reserved for mathematicians. But for people who will not do mathematics, certainly it is quite instructive to have the idea of a group acting on a vector space. So this is what the linear aspect should be.

The second aspect is at the 18–19 level: calculus, of course, including Cauchy's theory of complex functions of one complex variable, linear differential equations. In other words, it is Euler's programme (Euler was a master of this kind of analysis): compute! I have seen in many places in France people trying to teach that kind of thing in a very abstract way, introducing differential forms, integration of differential forms. For mathematicians, they will meet them later on. For those who go into physics they do not need to have that nonsense. But they should know very beautiful things about analysis such as asymptotic evaluation, the Laplace method for evaluating integral functions of large numbers, the method of steepest descent, the method of stationary phase, the Euler-Maclaurin formula, the Bernoulli numbers, the gamma functions, and so on. In other words, I want Euler's programme updated with a little bit of Cauchy. Well, if these same people who have heard a lot about either the Dedekind cuts or differential forms are confronted with an equation like

$$y'' + q(x)y = f(x),$$

where $q(x)$ is not a constant but a small function (so that you have a perturbation of a very elementary equation), what will they do? They are helpless. Helpless because this is not the equation in the book. It is a very easy elementary method. It does not need any kind of abstract mathematics to show how this kind of equation has a solution by the method of approximation which is very close to the unperturbed equation and that is the kind of thing which engineers will need. So this is the way I will like to be taught at that level. The main idea in analysis is not a matter of equalities, it is a matter of inequalities. You have to handle inequalities all the time.

What about the next level below, which is probably the most controvarsial of all? Well, I will follow the same idea except that, of course, now we are at a different level and we should think of what they should have at the next level. And so the first is that the linear part should be on the same basis but limited to what children know at that time, that is, 2 and 3 dimensions. In 2 and 3 dimensions, we have this beautiful fact that we are able to visualise. Well, when you talk about geometry to people at the level of n dimensions, you will need a little bit of training to visualise what a hyperplane is in 15 dimensions. You get a kind of intuition when you work with such things. You will get used to it. As far as Laurent can testify, when we start working with infinite dimensional spaces, we get a very good intuition. We have to be very careful at some point, of course. Normal geometric intuition does not always work but provided we are careful, we have a good intuition of how things work when we deal with subspaces of some very fancy functional spaces of infinite dimensions. So this kind of intuition can be gained by mathematicians through training. Perhaps you can get it from the finite dimensions, but it is not so easy. But here you should generally restrict yourself to 2 and 3 dimensions because then you have an enormous mass of intuition. And you can have, as I said earlier, every algebraic theorem as a geometric theorem and show it on the blackboard. So in that case, no algebraic theorem is without a geometric interpretation. Of course, this precludes something which I forgot to mention earlier. No other type of axiomatic definition of geometry. It is perfectly stupid to try to define geometry, Euclidian geometry, by taking the 23 axioms of Hilbert. It is sheer nonsense. We have axioms of a vector space, finite dimensional at that level, and that is quite enough.

For the other part, it is again a preparation towards the next stage, namely at that level, they have to know a little bit of calculus of one variable, to graph functions, to study integrals, and so on.

And finally we come to the curriculum which is, in France, generally the one which is most controversial at the present. What happens is that a lot of high school teachers or what we call "inspector-generals" who are former high school teachers turned into inspectors. They go and supervise the high school teachers. They know enough mathematics to prescribe the curriculum except that they learn their mathematics in university may be 50 years ago. So first of all, it is not the kind of mathematics which is taught nowadays. What they want to do and should have avoided at all costs at that level is an axiomatic approach. I am sure that children between 12 and 15 have no idea what an axiomatic approach is about and they do not see any reason why you should introduce that kind of thing. What they have to learn is what I call the physics of space.

Well, you teach them the physics of heat, the physics of resistance of materials, physics of weight, the physics of electricity and so on. You teach them to organise themselves in the world they are in. Through centuries of reflection on science we have been able to organise the extraordinary variety of phenomena around us into a series of coherent systems. That is the kind that should be taught without recourse to any axiomatic system. It is there. And we have this kind of thing which organises it into a coherent system. So we should teach them at that level the physics. of time. It is just as necessary that we should teach them the physics of space. It introduces them to the usual thing which happen in space: the plane, the line, the various transformations, rotation, translation, similarity mappings, and so forth. All these are purely experimental data. There should be no attempts at that level to axiomatise anything. They should just simply be introduced to the world around us, and in particular, the world of geometry, namely the physics of space. Well, that does not mean we should not introduce some logical argument, especially when at times it is very easy to do so. For instance, when you introduce the equilateral triangle and you draw the perpendicular bisectors of the 3 sides, and if your drawing is good, you will check that the lines meet in a single point. And now I am quite sure a good teacher will say to the children, "Now look here, this may look very surprising to you but it is bound to happen. And why is it bound to happen? The answer is that . . ." This is the first inkling of a logical argument which they learn is the basis of all mathematical arguments. That can be given and I am sure there are plenty of others which can be given. But it is stupid to try to give an argument to show, for instance, that opposite angles are equal. The children say, "Obvious," and they are quite right. It is a fact of the physics of space. They do not have to prove it. In the first example, they do not see it at all. And when they see it they are astonished. And it is natural that they should be given the proof.

In other words, at that level the various geometrical notions, arguments and transformations should be presented on a purely experimental basis. There will be no axiomatic system at all. And you see the transition at age 15, from stage 1 to 2, can then be made in a very nice way. When children arrive at that stage, they can be told the following things. "Now for 2 or 3 years you have been studying geometry, you have been acquainted with a lot of notions, a lot of properties. Now we are going to show you that all these properties which form a rather vast and not very organised system can be deduced from a very small number of them by purely logical argument." I think at that age children are able to follow the ideas and to see how powerful it is to be able to organise this welter of uncoordinated facts into a single system. That is where the axiomatic system comes in. For the usual vector spaces of 2 or 3 dimensions, give all the results in a purely logical way. Regarding the second type, at that level, I think it should be reduced to the technique of computation and evaluation of errors.

I think if the curriculum is based on that, it should certainly be quite conducive to forming not only mathematicians but other scientific people, some people with some tinge of science, and even those who have no interest in science at all. Thank you.