INTER-SCHOOL MATHEMATICAL COMPETITION 1979

The problems of the Inter-school Mathematical Competition 1979 are reproduced below.

Correct answers in Part A are marked with asterisks. Outlines of solutions to problems in Part B are included.

Part A

Friday, 16 March 1979 9.00 a.m. — 10.00 a.m.

1. How many divisors (including 1 and 1400) does the number 1400 have?
(a) 22; (b) 23; (c)* 24; (d) 25; (e) 26.

2. In Fig. 1, ABC and PQRS are squares. The ratio of the areas of PQRS and ABCD is approximately
(a) 0.58;
(b) 0.56;
(c)* 0.54;
(d) 0.52;
(e) 0.50.

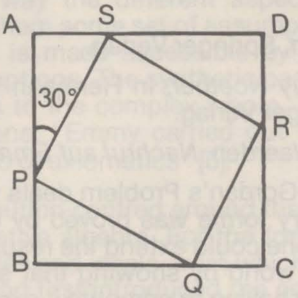


Fig. 1

3. If m and n are positive integers with $1 \leqslant m \leqslant 5$, $1 \leqslant n \leqslant 5$ and $m \neq n$, the number of (ordered) pairs (m, n) satisfying $m^n = n^m$ is
(a) 1; (b)* 2; (c) 3; (d) 4; (e) 5.

4 If $\binom{n}{k} = \dfrac{n(n-1)\ldots(n-k+1)}{1.2\ldots.k}$, then the sum

$$\binom{n}{1}1^2 + \binom{n}{2}2^2 + \binom{n}{3}3^2 + \ldots + \binom{n}{n}n^2 \text{ is equal to}$$

(a) $n(n+1)$; (b)* $n(n+1)2^{n-2}$; (c) $(n+1)2^{n-2}$; (d) $n2^{n-2}$; (e) $n^2 2^{n-2}$.

5. The coefficient of the term in x^{23} in the expansion of $(1 + x^5 + x^9)^{100}$ is
(a) 500,000; (b) 450,000; (c) 495,000; (d)* 485,100; (e) 454,200.

6. A heavy uniform chain is lightly placed on the two smooth inclined faces of a wedge PQR so that the chain is in one plane with the ends A and B at the same horizontal level and $\alpha \geqslant \beta$ (see Fig. 2). Which of the following statements is true?

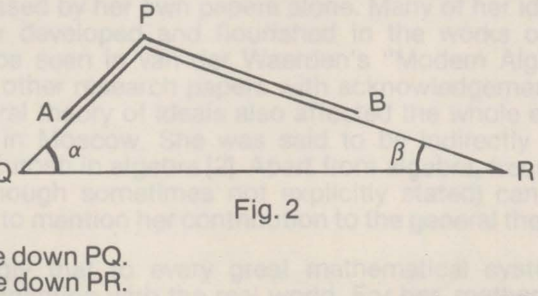


Fig. 2

(a) The chain will slide down PQ.
(b) The chain will slide down PR.
(c) The chain will rest in equilibrium if $\alpha = \beta$ but will slide down PQ if $\alpha > \beta$.
(d) The chain will rest in equilibrium if $\alpha = \beta$ but will slide down PR if $\alpha > \beta$.
(e)* The chain will rest in equilibrium.

7. The perpendicular distances (in cm) from an interior point of a given equilateral triangle to the sides of the triangle are 1, 3 and 2. The area (in cm^2) of the triangle is (a)* $6 + 4\sqrt{3}$; (b) $6 + 3\sqrt{3}$; (c) $4 + 2\sqrt{3}$; (d) $3 + 6\sqrt{3}$; (e) None of the above.

8. Four spheres each of unit radius lie on a rough horizontal table so that the centres of the spheres form a square of side 2 units. A fifth sphere of unit radius is placed on them so that it touches each of the spheres. The height of the centre of the fifth sphere above the table is
(a) 5; (b)* $1 + \sqrt{2}$; (c) $1 + \sqrt{3}$; (d) $\sqrt{6}$; (e) none of the above.

9. There are 6 students each of whom knows some gossip not known to the others. They communicate by telephone and whenever one student calls another, they tell each other all they know at that time. The minimum number of telephone calls that must be made so that every student knows all the gossip is
(a) 6; (b)* 8; (c) 10; (d) 15; (e) none of the above.

10. Let m and n be positive integers with $m \geqslant n > 1$. In how many ways can m be written as a sum of n positive integers $m = u_1 + u_2 + \ldots + u_n$ where u_i is a positive integer for $1 \leqslant i \leqslant n$? (Two such sums are considered to be distinct if they differ either in the terms u_i or their order.)

(a) $\binom{m}{n-1} + 1$; (b) $\binom{m}{n} - 1$; (c) $\binom{m+1}{n+1}$; (d)* $\binom{m-1}{n-1}$; (e) none of the above.

Part B

Friday, 16 March 1979 — 10.00 a.m. — 12.00 noon

1. Let ABC be a right-angled triangle and CH the altitude on the hypotenuse AB. show that the sum of the radii of the inscribed circles of triangles ABC, HCA and HCB is equal to the length of the altitude CH.

2. If a trlangle of sides a, b, c, has area Δ, show that
$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\,\Delta.$$
Under what conditions does equality hold?

3. Consider the following array of numbers:

	column 1	column 2	...	column n
row 1	a_{11}	a_{12}	...	a_{1n}
row 2	a_{21}	a_{22}	...	a_{2n}
.	.	.		.
.	.	.		.
.	.	.		.
row n	a_{n1}	a_{n2}	...	a_{nn}

For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, a_{ij} is called an entry and is equal to 0 or 1. Suppose that whenever an entry is 0, the sum of all the entries occurring in the row or column containing that zero entry is at least n. By considering the row or column with the smallest sum, show that the sum of all the entries in the array is at least $n^2/2$.

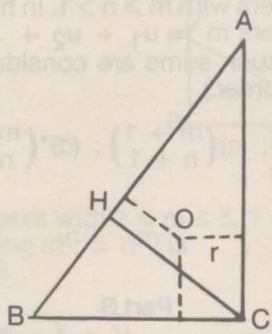
4. Let $a_1, 1_2, \ldots, a_{n+1}$ be integers such that $1 \leqslant a_1 < a_2 < \ldots < a_{n+1} \leqslant 2n$. By considering the largest odd divisors of the a_i, show that a_i is a divisor of a_j for some i and j with $i \neq j$.

5. Let f(x) be a polynomial in x with integral coefficients. If the rational number p/q, where p and q are relatively prime, is a root of the equation $f(x) = 0$, show, by expressing f(x) as a polynomial in $x - 1$ or otherwise, that $p - q$ is a divisor of f(1).

6. Show that if $a + \sin(\pi/18)$, then $8a^3 - 6a + 1 = 0$. Deduce that $\sin(\pi/18)$ is irrational.

Solutions to Part B

1.



Denote by a, b, c the sides of triangle ABC according to the usual convention. Let r be the inradius of triangle ABC. We first show that

$$2r = a + b - c.$$

Let O be the incentre of triangle ABC. By considering the areas of triangles ABC, OAB, OBC, OCA, we have

$$\frac{1}{2}ab = \frac{1}{2}r(a + b + c)$$

Hence
$$\begin{aligned} r &= ab/(a + b + c) \\ &= ab(a + b - c)/((a + b)^2 - c^2) \\ &= \frac{1}{2}(a + b - c), \end{aligned}$$

since $a^2 + b^2 = c^2$.

Now let r_1, r_2 be the inradii of triangles HCA, HCB respectively. Let $AH = c_1$, $BH = c_2$, $CH = h$. Then, from above, we have

$$r_1 = \frac{1}{2}(c_1 + h - b), \quad r_2 = \frac{1}{2}(c_2 + h - a).$$

Hence $r + r_1 + r_2 = h$.

2. With the usual notations, $\Delta = \frac{1}{2}ab\sin C$, and so

$$\begin{aligned}4\Delta^2 &= a^2b^2\sin^2 C)\\ &= a^2b^2(1 - \cos^2 C)\\ &= a^2b^2 - \frac{1}{4}(a^2 + b^2 - c^2)^2\\ &= (a^2b^2 + a^2c^2 + b^2c^2) - \frac{1}{4}(a^2 + b^2 + c^2)^2.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{1}{3}(a^2 + b^2 + c^2) - 16\Delta^2 &= \frac{4}{3}[(a^4 + b^4 + c^4) - (a^2b^2 + b^2c^2 + c^2a^2)]\\ &= \frac{2}{3}[(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2]\\ &\geqslant 0\end{aligned}$$

3. Since the entries are non-negative integers, it is sufficient to assume that each entry is either 0 or 1. In general, the required sum will be greater than or equal to the sum of entries of the above type. Let the sum of the ith row be the smallest possible sum among all rows and columns (same argument may be used if it is a column that gives the smallest sum). Let this sum be s. Then there are at most s nonzero entries in the ith row, and so there are at most n – s zeros in the ith row. Consider all those columns having zeros in the ith row. There are at most n – s such columns. By the condition given, the sum of each such column is at least n-s. The remaining s columns each has a sum of at least s (since s is the smallest row or column sum). So the sum of all entries is at least $(n - s)^2 + s^2 = \frac{1}{2}n^2 + 2(s - \frac{1}{2}n)^2 \geqslant \frac{1}{2}n^2$.

4. Let b_i be the largest add divisor of a_i. The integers $b_1, b_2, \ldots, b_{n+1}$ cannot be all distinct because there are only n odd integers m such that $1 \leqslant m < 2n$. Thus $b_i = b_j$ for some integers, i, j with $1 \leqslant i < j \leqslant n + 1$. Let $a_i = 2^{\alpha_i}b_i$, $a_j = 2^{\alpha_j}b_j$. Since $a_i < a_j$, we have $\alpha_i < \alpha_j$, and hence a_i is a divisor of a_j.

5. Let $f(x) = a_o(x - 1)^n + a_1(x - 1)^{n-1} + \ldots + a_{n-1}(x - 1) + a_n$. Then the a_i are integers. Since $f(p/q) = 0$, we have

$$a_o(\frac{p}{q} - 1)^n + \ldots + a_{n-1}(\frac{p}{q} - 1) + a_n = 0,$$

$$a_o(p - q)^n + \ldots + a_{n-1}(p - q)q^{n-1} + a_nq^n = 0.$$

This shows that $p - q$ is a divisor of a_nq^n. But $p - q$ and q are relatively prime, and so $p - q$ divides $a_n = f(1)$.

6. That $\sin(\pi/18)$ satisfies $8a^3 - 6a + 1 = 0$ follows from the identity $\sin 3x = 3\sin x - 4\sin^3 x$ and the fact that $\sin(\pi/6) = \frac{1}{2}$. The only possible rational solutions of the above cubic equation are $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}$. Substitutions of these values show that the equation is not satisfied. Hence the above cubic equation has no rational solutions, and so $\sin(\pi/18)$ must be irrational.