PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Members are reminded that although no prizes are awarded to the contribution of problems, interesting problems at secondary school level are most welcome. Problems or solutions should be sent to Dr. K.N. Cheng, Department of Mathematics, University of Singapore, Singapore 10.

P. 9/78. In a certain chess championship between two players A and B, the title goes to the one who first scores six wins; draws do not count. The probability that A wins a game is $\frac{1}{4}$ and the probability that he draws is $\frac{1}{2}$. If A now leads by five wins to B's four wins, what is the probability that A wins the title eventually? Assume that the games have independent outcomes.

(Tay Yong Chiang)

P 10/78. The following "proof" that the alternating group A_5 of degree five is simple appears in *The Fascination of Groups* by F.J. Budden: A_5 has one identity, 24 5-cycles, 20 3-cycles, and 15 double transpositions, making a total of 60 elements. Now we know that if a normal subgroup contains a particular element, then it contains every one of its conjugates. It follows that the order of a normal subgroup of A_5 must be of the form

$$1 + 24n_1 + 20n_2 + 15n_3, \text{ where } n_1, n_2, n_3 \in \{0, 1\}.$$

Also this number must be a factor of 60, the order A_5. This is only possible if $n_1 = n_2 = n_3 = 0$. Hence A_5 has only trivial normal subgroups; i.e. A_5 is simple.

Find and correct the error in this "proof".

<div style="text-align: right">(via K.M. Chan)</div>

*P 11/78. Find the total number of images formed by placing a point object A between the reflecting sides of two plane mirrors which intersect at an acute angle α. Assume that A is at a perpendicular distance d from the line of intersection L of the two mirrors, with OA making an angle β with one of the mirrors, O being the foot of perpendicular from A to L (see diagram below).
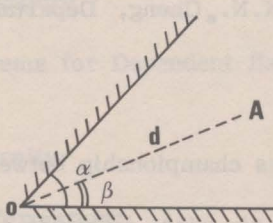
<div style="text-align: right">(Y.K. Leong)</div>

Solutions to P6 - P 8/78.

*P 6/78 Let T, T' be two linear transformations of the three-dimensional Euclidean space V into itself. Let A, A' be the matrices of T, T' respectively with respect to some fixed rectangular axes Oxyz with origin O. Further let A' be the transpose of A. Prove that if P, Q, R are points of V such that T: P $\longrightarrow$ Q, T': P $\longrightarrow$ R, then the line OP is perpendicular to the line QR.

<div style="text-align: right">(Via Ho Soo Thong)</div>

Solution by Tay Yong Chiang:

We first note that a linear transformation of V represented by a non-zero skew-symmetric matrix

$$\begin{pmatrix} o & a & b \\ -a & o & c \\ -b & -c & o \end{pmatrix} \text{ say,}$$

maps a point $L = (x, y, z)$ of V to a point $L' = (x', y', z')$ of V such that the line OL is perpendicular to the line OL'. For

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} o & a & b \\ -a & o & c \\ -b & -c & o \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + bz \\ -ax + cz \\ -bx - cy \end{pmatrix},$$

and one verifies easily that $\overrightarrow{OL}. \overrightarrow{OL'} = O$, hence the assertion.

Now let $P = (x_p, y_p, z_p)$, $Q = (x_q, y_q, z_q)$ and $R = (x_r, y_r, z_r)$.

Then

$$A \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} x_q \\ y_q \\ z_q \end{pmatrix},$$

$$A' \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} x_r \\ y_r \\ z_r \end{pmatrix}$$

and

$$(A - A') \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} x_q - x_r \\ y_q - y_r \\ z_q - z_r \end{pmatrix}. \qquad \ldots. (1)$$

Set $S = (x_q - x_r, y_q - y_r, z_q - z_r)$. Then the line OS is parallel to QR. Now since Q and R are distinct points of V and $A' = A^T$, the transpose of A, $A - A'$ is a non-zero skew symmetric matrix. From (1) and the observation earlier on, it follows that OP is perpendicular to OS, and hence to QR, since OS is parallel to QR.

Alternative solution by Goh Koon Shim:

Set $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $P = (x_p, y_p, z_p)$ $Q = (x_q, y_q, z_q)$ and $R = (x_r, y_r, z_r)$.

Then

$$\begin{pmatrix} x_q \\ y_q \\ z_q \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}$$

$$= \begin{pmatrix} ax_p + by_p + cz_p \\ dx_p + ey_p + fz_p \\ gx_p + hy_p + iz_p \end{pmatrix}$$

and

$$\begin{pmatrix} xr \\ yr \\ zr \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}$$

$$= \begin{pmatrix} ax_p + dy_p + gz_p \\ bx_p + ey_p + hz_p \\ cx_p + fy_p + iz_p \end{pmatrix}$$

Now the direction cosines (l, m, n) of $\overrightarrow{OP}$ are $\left(\dfrac{x_p}{r}, \dfrac{y_p}{r}, \dfrac{z_p}{r} \right)$ where $r = \sqrt{x_p^2 + y_p^2 + z_p^2}$; and the direction cosines (l', m', n') of $\overrightarrow{QR}$ are $\left(\dfrac{x_r - x_q}{t}, \dfrac{y_r - y_q}{t}, \dfrac{z_r - z_q}{t} \right)$, where $t = \sqrt{\{(x_r - x_q)^2 + (y_r - y_q)^2 + (z_r - z_q)^2\}}$. One verifies easily now that $ll' + mm' + nn' = 0$, which implies immediately that OP is perpendicular to QR.

*P 7/78. Given $\triangle ABC$. Let L, M be points on AB, BC respectively such that $AL : LB = BM : MC = k$, where k is some finite non-zero real number. If P, Q are points on AB, BC respectively such that PQ intersects LM at T with $PT : TQ = k$, show that $AP : PB = BQ : QC = LT : TM$.

Solution:

Given AL : LB = MC = PT : TQ = k. Consider first of all the case when k is positive.
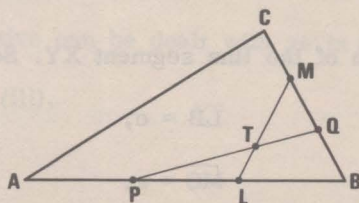We may refer to Fig. (i).

Fig. 1.

Solution (1) by Teo Tat Khai and Tay Yong Chiang:

Let $\overrightarrow{BA} = \underset{\sim}{c}$, $\overrightarrow{BC} = \underset{\sim}{a}$, $\overrightarrow{BP} = \lambda \underset{\sim}{c}$ and $\overrightarrow{BQ} = \mu \underset{\sim}{a}$, where λ, μ are some finite real numbers.
Further let $\overrightarrow{LT} : \overrightarrow{TM} = h$,
By considering $\triangle BLM$ and $\triangle BPQ$ respectively, we get

$$\overrightarrow{BT} = \frac{\overrightarrow{BL} + h \overrightarrow{BM}}{1 + h} = \frac{\overrightarrow{BP} + k\overrightarrow{BQ}}{1 + k} \qquad \cdots \text{ (I)}$$

$$\text{Now } \overrightarrow{BM} = \frac{k}{1 + k} \underset{\sim}{a}, \qquad \cdots \text{(II)}$$

$$\overrightarrow{BL} = \frac{1}{1 + k} \underset{\sim}{c}.$$

From (I) and (II) we get

$$\frac{1}{1 + k} \left[\frac{1}{1 + h} - \lambda \right] \underset{\sim}{c} = \frac{1}{1 + k} \left[k\mu - \frac{hk}{1 + h} \right] \underset{\sim}{a}$$

(Note that $1 + h \neq 0 \neq 1 + k$). Since $\underset{\sim}{a}$ is not parallel to $\underset{\sim}{c}$, this is only possible if

$$\frac{1}{1 + h} - \lambda = 0 = k\mu - \frac{hk}{1 + h}$$

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From which we have $\lambda = \dfrac{1}{1 + h}$ and $\mu = \dfrac{h}{1 + h}$. This gives AP : PB = h = BQ : QC = LT : TM.

Solution (2) by **Chan Sing Chun**:

Denote by $\overline{XY}$ the length of the line segment XY. Set

$$\overline{LB} = c,$$

$$\overline{MC} = a,$$

$$\overline{TQ} = b.$$

Then $\overline{AL} = kc$, $\overline{BM} = ka$, $\overline{PT} = kb$. We may further set $\overline{PL} = nc$ and $\overline{QM} = ma$, where n, m are some finite non-negative real numbers.

Consider $\triangle BPQ$ with transversal LTM. By Menelaus Theorem,

$$\frac{\overline{BL}}{\overline{LP}} \cdot \frac{\overline{QM}}{\overline{MB}} \cdot \frac{\overline{PT}}{\overline{TQ}} = 1,$$

so

$$\frac{c}{nc} \cdot \frac{ma}{ka} \cdot \frac{kb}{b} = 1,$$

and

$$m = n$$

We get

$$AP : PB = \overline{AP} : \overline{PB} = \frac{kc - mc}{mc + c} = \frac{k - m}{m + 1},$$

and

$$BQ : QC = \overline{BQ} : \overline{QC} = \frac{ka - ma}{ma + a} = \frac{k - m}{m + 1} .$$

Now consider $\triangle BLM$ with transversal PTQ. By Menelaus Theorem,

$$\frac{\overline{BP}}{\overline{PL}} \cdot \frac{\overline{LT}}{\overline{TM}} \cdot \frac{\overline{MQ}}{\overline{QB}} = 1,$$

so
$$\frac{c + mc}{mc} \cdot \frac{\overline{LT}}{\overline{TM}} \cdot \frac{ma}{ka - ma} = 1,$$

and
$$\frac{\overline{LT}}{\overline{TM}} = \frac{k - m}{k + 1}$$

Hence LT : TM = $\overline{LT}$: $\overline{TM}$ = $\frac{k - m}{k + 1}$, and AP : PB = BQ : QC = LT : TM.

The case when k is negative can be dealt with as in solution (1) or solution (2). One may refer to Fig. (ii) or Fig. (iii).
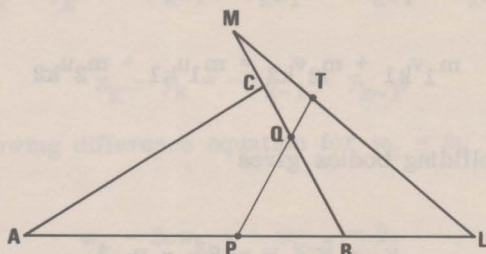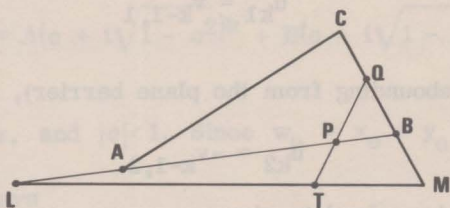

Fig. (ii)

Fig. (iii)

*P 8/78. A sphere of mass m_1 moves with uniform velocity V on a smooth horizontal floor and impinges upon a stationary sphere of mass m_2, which then moves on the floor until it hits a fixed vertical plane barrier. In order to allow as many impacts as possible, the barrier is shifted parallel to itself throw some distance away from the rebounding sphere

after each impact and kept fixed before the next impact. Assuming that all collisions are direct and perfectly elastic, find the total number of collisions between the two spheres.

Solution by Y.K. Leong

Let the velocities of the spheres of masses m_1 and m_2 before the kth impact be u_{k1} and u_{k2} respectively, and the corresponding velocities after the kth impact be v_{k1} and v_{k2}. Thus $u_{11} = V$, $u_{12} = 0$. The direction of the initial velocity V is taken to be positive.

Conservation of momentum gives

$$m_1 v_{k1} + m_2 v_{k2} = m_1 u_{k1} + m_2 u_{k2} \qquad \dots \text{(1)}$$

Newton's law for colliding bodies gives

$$v_{k2} - v_{k1} = -(u_{k2} - u_{k1}) \qquad \dots \text{(2)}$$

For the mass m_1, we have

$$u_{k1} = v_{k-1, 1} \qquad \dots \text{(3)}$$

For the mass m_2 (after rebounding from the plane barrier), we have

$$u_{k2} = -v_{k-1, 2} \qquad \dots \text{(4)}$$

Equations (1) and (2) may be solved to give

$$\begin{pmatrix} v_{k1} \\ v_{k2} \end{pmatrix} = \begin{pmatrix} c & 1 - c \\ 1 + c & - c \end{pmatrix} \begin{pmatrix} u_{k1} \\ u_{k2} \end{pmatrix} \qquad \dots \text{(5)}$$

where $c = (m_1 - m_2)/(m_1 + m_2)$. Using (3), (4), we write (5) as

$$\begin{pmatrix} v_{k1} \\ v_{k2} \end{pmatrix} = \begin{pmatrix} c & c - 1 \\ c + 1 & c \end{pmatrix} \begin{pmatrix} v_{k-1, 1} \\ v_{k-1, 2} \end{pmatrix} \qquad \dots \text{(6)}$$

Assuming that the kth collision has taken place, the $(k + 1)$ collision with occur if and only if $v_{k1} - (- v_{k2}) > 0$;

that is
$$v_{k1} + v_{k2} > 0 \qquad \qquad \cdots \cdots (7)$$

For simplicity, write $x_k = v_{k1}$, $y_k = v_{k2}$, where $x_0 = V$, $y_0 = 0$. From (6), we have

$$x_k = cx_{k-1} + (c - 1)y_{k-1},$$

$$y_k = (c + 1)x_{k-1} + cy_{k-1}.$$

Then

$$x_k + y_k = 2c(x_{k-1} + y_{k-1}) + (x_{k-1} - y_{k-1}), \qquad \cdots \cdots (8)$$

$$x_k - y_k = -(x_{k-1} + y_{k-1}) \qquad \cdots \cdots (9)$$

Hence we obtain the following difference equation for $w_k = x_k + y_k$:

$$w_k - 2cw_{k-1} + w_{k-2} = 0. \qquad \cdots \cdots (10)$$

The general solution of (10) is

$$w_k = A\left(c + i\sqrt{1 - c^2}\right)^k + B\left(c - i\sqrt{1 - c^2}\right)^k,$$

where A, B are constants, and $|c| < 1$. Since $w_0 = x_0 + y_0 = V$, and $w_1 = x_1 + y_1 = (1 + 2c)V$ (from (8)), we have

$$A + B = V,$$

$$A\left(c + i\sqrt{1 - c^2}\right) + B\left(c - i\sqrt{1 - c^2}\right) = (1 + 2c)V.$$

Hence,

$$A = \tfrac{1}{2}V\left(1 - i\sqrt{\frac{1 + c}{1 - c}}\right),$$

$$B = \tfrac{1}{2}V\left(1 + i\sqrt{\frac{1 + c}{1 - c}}\right).$$

Thus,

$$w_k = \tfrac{1}{2}V\left\{(c + i\sqrt{1 - c^2})k + (c - i\sqrt{1 - c^2})k\right\}$$

$$-\tfrac{1}{2}Vi\sqrt{\frac{1 + c}{1 - c}}\left\{(c + i\sqrt{1 - c^2})k - (c - i\sqrt{1 - c^2})\,k\right\}.$$

This expression may be simplified by putting $c = \cos \alpha$, where $0 < \alpha < \pi$, to yield

$$w_k = V(\cos k\alpha + \cot \tfrac{1}{2}\alpha \sin k\alpha),$$

or

$$w_k = V \sin (k + \tfrac{1}{2})\alpha \,/\, \sin \tfrac{1}{2}\alpha . \qquad \qquad \text{.... (11)}$$

Here use is made of de Moivre's formula $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$.

As remarked before (see (7)) there will be another collision between the spheres after the kth collision if and only if $w_k > 0$. Now if $k = k_0$ is the largest integer such that $(k_0 + \tfrac{1}{2})\alpha < \pi$, then it follows from (11) that $w_{k_0} > 0$ and $w_{k_0+1} \leqslant 0$. Hence there will be a total of $(1 + k_0)$ collisions; or, the number of collisions is equal to the largest integer less than $\tfrac{1}{2} + (\pi/\alpha)$, where $\cos \alpha = (m_1 - m_2)/(m_1 + m_2)$ and $0 < \alpha < \pi$

Tay Yong Chiang and Teo Tat Khai have each been awarded a book-voucher prize of \$5 for the solutions of Problem 6 and Problem 7 respectively.